Appendix 12: Vector Calculus

The del operator

$$\nabla = \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z} \quad (A12.1)$$

The divergence

$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \quad (A12.2)$$

The gradient

$$\nabla \phi = \frac{\partial \phi}{\partial x} \hat{e}_x + \frac{\partial \phi}{\partial y} \hat{e}_y + \frac{\partial \phi}{\partial z} \hat{e}_z \quad (A12.3)$$

The Laplacian

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \quad (A12.4)$$

The Laplacian in cylindrical coordinates

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z \quad (A12.6)$$

$$\nabla^2 \Psi = \frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r} \frac{\partial \Psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \theta^2} + \frac{\partial^2 \Psi}{\partial z^2}$$

The Laplacian in spherical coordinates

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta \quad (A12.7)$$

$$\nabla^2 \Psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial \Psi}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Psi}{\partial \phi^2}$$

The vector product
\[ \nabla \times \vec{\varepsilon} = \begin{vmatrix} \varepsilon_x & \varepsilon_y & \varepsilon_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varepsilon_x & \varepsilon_y & \varepsilon_z \end{vmatrix} \]  

\[ = \left( \frac{\partial \varepsilon_z}{\partial y} - \frac{\partial \varepsilon_y}{\partial z} \right) \varepsilon_x - \left( \frac{\partial \varepsilon_x}{\partial z} - \frac{\partial \varepsilon_z}{\partial x} \right) \varepsilon_y + \left( \frac{\partial \varepsilon_y}{\partial x} - \frac{\partial \varepsilon_x}{\partial y} \right) \varepsilon_z \]  

(A12.5)