The One Dimensional Schrödinger Equation

\[ i\hbar \frac{d\psi(x, t)}{dt} = -\frac{\hbar^2}{2\mu} \frac{\partial^2 \psi(x, t)}{dx^2} + V(x)\psi(x, t) \]

The 1-D Paraxial Wave Equation

\[ ik_0 \frac{d\phi(x, z)}{dz} = -\frac{1}{2} \frac{\partial^2 \phi(x, z)}{dx^2} - k^2(x)\phi(x, z)/2 \]
The 1-D Time Independent Schrödinger Equation

\[ \frac{\partial^2 \psi(x)}{dx^2} + \frac{2\mu}{\hbar^2} (E - V(x)) \psi(x) = 0 \]

The 1-D Paraxial Wave Eigen Equation

\[ \frac{\partial^2 \phi(x)}{dx^2} + \left( k^2(x) - 2ik_0\beta \right) \phi(x) = 0 \]
The 1-D Schrödinger Equation, with $V(x) = V_0$ inside

$$\frac{\partial^2 \psi(x)}{dx^2} + \frac{2\mu}{\hbar^2} (E - V_0) \psi(x) = 0$$

and its solutions

$$\psi_n(x) = A_n \cos k_n x + B_n \sin k_n x$$

$$k_n = \sqrt{\frac{2\mu (E_n - V_0)}{\hbar^2}}$$
The 1-D Schrödinger Equation, with $V(x) = V_0$ outside

\[ \frac{\partial^2 \psi(x)}{dx^2} + \frac{2\mu}{\hbar^2} (V_0 - E) \psi(x) = 0 \]

and its solutions

\[ \psi_n(x) = A_n \exp(-\kappa_n x) + B_n \exp(\kappa_n x) \]
\[ \kappa_n = \sqrt{\frac{2\mu (V_0 - E_n)}{\hbar^2}} \]
The 1-D Schrödinger Equation, with $V(x) = \mu \omega^2 x^2 / 2$

$$\frac{\partial^2 \psi(x)}{dx^2} + \frac{2\mu}{\hbar^2} (E - V(x)) \psi(x) = 0$$

and its solutions

$$\psi_n(\xi) = A_n h_n(\xi) \exp(-\xi^2 / 2)$$

$$\xi = \sqrt{\frac{\mu \omega}{\hbar}} x$$
The solution of the 1-D Schrödinger Equation is

$$|\psi(t)\rangle = \sum_n \int dx \langle \psi_0 |x\rangle \langle x|n\rangle |n\rangle \exp(-iE_nt/\hbar)$$

which is equivalent to

$$|\psi(t)\rangle = \sum_n \langle \psi_0 |n\rangle |n\rangle \exp(-iE_nt/\hbar)$$
The Two Dimensional Schrödinger Equation

\[ i\hbar \frac{d\psi(x, y, t)}{dt} = -\frac{\hbar^2}{2\mu} \left( \frac{\partial^2 \psi(x, y, t)}{dx^2} + \frac{\partial^2 \psi(x, y, t)}{dy^2} \right) + V(x, y)\psi(x, y, t) \]

The 2-D Paraxial Wave Equation

\[ ik_0 \frac{d\phi(x, y, z)}{dz} = -\frac{1}{2} \left( \frac{\partial^2 \phi(x, y, z)}{dx^2} + \frac{\partial^2 \phi(x, y, z)}{dy^2} \right) - k^2(x)\phi(x, y, z)/2 \]
The 2-D Time Independent Schrödinger Equation

\[
\frac{\partial^2 \psi(x, y)}{dx^2} + \frac{\partial^2 \psi(x, y)}{dy^2} + \frac{2\mu}{\hbar^2}(E - V(x, y))\psi(x, y) = 0
\]

The 2-D Paraxial Wave Eigen Equation

\[
\frac{\partial^2 \phi(x, y)}{dx^2} + \frac{\partial^2 \phi(x, y)}{dy^2} + \left(k^2(x, y) - 2ik_0\beta\right)\phi(x, y) = 0
\]
The 2-D Schrödinger Equation, with \( V(x, y) = V_0 \) inside

\[
\frac{\partial^2 \psi(x, y)}{dx^2} + \frac{\partial^2 \psi(x, y)}{dy^2} + \frac{2\mu}{\hbar^2} (E - V_0) \psi(x, y) = 0
\]

and its solutions

\[
\psi_{lm}(x, y) = (A_l \cos k_xx + B_l \sin k_xx)(C_m \cos k_yy + D_m \sin k_yy)
\]

\[
k_x^2 + k_y^2 = \frac{2\mu (E_{lm} - V_0)}{\hbar^2}
\]
The 2-D Schrödinger Equation, with \( V(x, y) = V_0 \) outside

\[
\frac{\partial^2 \psi(x, y)}{dx^2} + \frac{\partial^2 \psi(x, y)}{dy^2} + \frac{2\mu}{\hbar^2} (V_0 - E) \psi(x, y) = 0
\]

and its solutions

\[
\psi_{lm}(x, y) = (A_l \exp(-\kappa_xx) + B_l \exp(\kappa_xx))(C_m \exp(-\kappa_yy) + D_m \exp(\kappa_yy))
\]

\[
\kappa_x^2 + \kappa_y^2 = \frac{2\mu (V_0 - E_{lm})}{\hbar^2}
\]
The 2-D Schrödinger Equation, with $V(r, \theta) = V_0$ inside

$$\frac{\partial^2 \psi(r, \theta)}{dr^2} + \frac{1}{r} \frac{\partial \psi(r, \theta)}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi(r, \theta)}{d\theta^2} + \frac{2\mu}{\hbar^2} (E - V_0) \psi(r, \theta) = 0$$

and its solutions

$$\psi_{lm}(r, \theta) = (AJ_m(kr) + BY_m(kr)) \exp(i m \theta)$$

$$k_{lm} = \sqrt{\frac{2\mu (E_{lm} - V_0)}{\hbar^2}}$$
The 2-D Schrödinger Equation, with $V(r, \theta) = V_0$ outside

$$\frac{\partial^2 \psi(r, \theta)}{dr^2} + \frac{1}{r} \frac{\partial \psi(r, \theta)}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi(r, \theta)}{d\theta^2} + \frac{2\mu}{\hbar^2} (V_0 - E) \psi(r, \theta) = 0$$

and its solutions

$$\psi_{lm}(r, \theta) = (A I_m(\kappa_{lm} r) + B K_m(\kappa_{lm} r)) \exp(im\theta)$$

$$\kappa_{lm} = \sqrt{\frac{2\mu (V_0 - E_{lm})}{\hbar^2}}$$
The 2-D Schrödinger Equation, with 
\[ V(x, y) = \mu (\omega_x^2 x^2 + \omega_y^2 y^2)/2 \]

\[ \frac{\partial^2 \psi(x, y)}{dx^2} + \frac{\partial^2 \psi(x, y)}{dy^2} + \frac{2 \mu}{\hbar^2} (E - V(x, y)) \psi(x, y) = 0 \]

and its solutions

\[
\psi_{lm}(x, y) = A_{lm} h_l(\xi) \exp(-\xi^2/2) h_m(\eta) \exp(-\eta^2/2)
\]

\[
\xi = \sqrt{\frac{\mu \omega_x}{\hbar}} x
\]

\[
\eta = \sqrt{\frac{\mu \omega_y}{\hbar}} y
\]
The 2-D Schrödinger Equation, with $V(r) = \mu \omega^2 r^2/2$

$$\frac{\partial^2 \psi(r, \theta)}{dr^2} + \frac{2\mu}{\hbar^2} (E - V(r)) \psi(r, \theta) = 0$$

and its solutions

$$\psi_{lm}(\xi, \eta) = A_{lm} \exp(-\rho^2/2) L_l^m(\rho) \exp(im\theta)$$

$$\rho = \sqrt{\frac{\mu \omega}{\hbar}} r$$
The solution of the 2-D Schrödinger Equation is

\[ |\psi(t)\rangle = \sum_{lm} \int dx \, dy \, \langle \psi_0 | x, y \rangle \langle x, y | lm \rangle |lm\rangle \exp(-iE_{lm}t/\hbar) \]

which is equivalent to the matrix equation

\[ |\psi(t)\rangle = \sum_{lm} \langle \psi_0 | lm \rangle |lm\rangle \exp(-iE_{lm}t/\hbar) \]
\[ = \sum_{n} \langle \psi_0 | n \rangle |n\rangle \exp(-iE_{n}t/\hbar) \]
The 3-D Schrödinger Equation, with $V(x, y, z) = V_0$ inside

$$\frac{\partial^2 \psi(x, y, z)}{dx^2} + \frac{\partial^2 \psi(x, y, z)}{dy^2} + \frac{\partial^2 \psi(x, y, z)}{dz^2} + \frac{2\mu}{\hbar^2} (E - V_0) \psi(x, y, z) = 0$$

and its solutions

$$\psi_{nlm}(x, y) = (A_l \cos k_xx + B_l \sin k_xx)$$

$$\quad \quad \quad \quad \quad \quad \quad \quad (C_m \cos k_yy + D_m \sin k_yy)$$

$$\quad \quad \quad \quad \quad \quad \quad \quad (E_n \cos k_zz + F_n \sin k_zz)$$

$$k_x^2 + k_y^2 + k_z^2 = \frac{2\mu (E_{nlm} - V_0)}{\hbar^2}$$
The 3-D Schrödinger Equation, with \( V(x, y, z) = V_0 \) outside

\[
\frac{\partial^2 \psi(x, y, z)}{dx^2} + \frac{\partial^2 \psi(x, y, z)}{dy^2} + \frac{\partial^2 \psi(x, y, z)}{dz^2} + \frac{2\mu}{\hbar^2} \left( V_0 - E \right) \psi(x, y, z) = 0
\]

and its solutions

\[
\psi_{nlm}(x, y, z) = (A_l \exp(-\kappa_x x) + B_l \exp(\kappa_x x)) \\
(C_m \exp(-\kappa_y y) + D_m \exp(\kappa_y y)) \\
(E_n \exp(-\kappa_z z) + F_n \exp(\kappa_z z))
\]

\[
\kappa_x^2 + \kappa_y^2 + \kappa_z^2 = \frac{2\mu(V_0 - E_{nlm})}{\hbar^2}
\]
The 3-D Schrödinger Equation, with $V(r, \theta, \phi) = V_0$ inside

\[
\begin{aligned}
\frac{\partial^2 \psi(r, \theta, \phi)}{dr^2} + & \frac{2}{r} \frac{\partial \psi(r, \theta, \phi)}{\partial r} + \\
\frac{1}{r^2} \left( \frac{\partial^2 \psi(r, \theta, \phi)}{d\theta^2} + & \cot(\theta) \frac{\partial \psi(r, \theta, \phi)}{\partial \theta} + \frac{1}{\sin^2(\theta)} \frac{\partial^2 \psi(r, \theta, \phi)}{d\phi^2} \right) + \\
+ & \frac{2\mu}{\hbar^2} (E - V_0) \psi(r, \theta, \phi) = 0
\end{aligned}
\]

and its solutions

\[
\psi_{nlm}(r, \theta, \phi) = (A_{jl}(k_{nlm}r) + B_{jl}(k_{nlm}r))Y_{lm}(\theta, \phi)
\]

\[
k = \sqrt{\frac{2\mu (E_{nlm} - V_0)}{\hbar^2}}
\]
The 3-D Schrödinger Equation, with $V(r, \theta, \phi) = V_0$ inside

$$\frac{\partial^2 \psi(r, \theta, \phi)}{dr^2} + \frac{2}{r} \frac{\partial \psi(r, \theta, \phi)}{\partial r} + \frac{1}{r^2} \left( \frac{\partial^2 \psi(r, \theta, \phi)}{d\theta^2} + \cot(\theta) \frac{\partial \psi(r, \theta, \phi)}{\partial \theta, \phi} + \frac{1}{\sin^2(\theta)} \frac{\partial^2 \psi(r, \theta, \phi)}{d\phi^2} \right) + \frac{2\mu}{\hbar^2} (V_0 - E) \psi(r, \theta, \phi) = 0$$

and its solutions

$$\psi_{nlm}(r, \theta, \phi) = (A_i l(k_{nlm} r) + B k_l(k_{nlm} r)) Y_{lm}(\theta, \phi)$$

$$k_{nlm} = \sqrt{\frac{2\mu (V_0 - E_{nlm})}{\hbar^2}}$$
The 3-D Schrödinger Equation, with
\[ V(x, y, z) = \mu (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)/2 \]

\[ \frac{\partial^2 \psi(x, y, z)}{dx^2} + \frac{\partial^2 \psi(x, y, z)}{dy^2} + \frac{\partial^2 \psi(x, y, z)}{dz^2} + \frac{2\mu}{\hbar^2} (V(x, y, z) - E) \psi(x, y, z) = 0 \]

and its solutions

\[ \psi_{nlm}(x, y, z) = A_{nlm} h_l(\xi) \exp(-\xi^2/2) h_m(\eta) \exp(-\eta^2/2) h_n(\zeta) \exp(-\zeta^2/2) \]

\[ \xi = \sqrt{\frac{\mu \omega_x}{\hbar}} x \]
\[ \eta = \sqrt{\frac{\mu \omega_y}{\hbar}} y \]
\[ \zeta = \sqrt{\frac{\mu \omega_z}{\hbar}} z \]
The 3-D Schrödinger Equation, with
\[ V(r, \theta, \phi) = V(r) = \mu \omega^2 r^2 / 2 \]

\[
\frac{\partial^2 \psi(r, \theta, \phi)}{dr^2} + \frac{2}{r} \frac{\partial \psi(r, \theta, \phi)}{\partial r} + \\
\frac{1}{r^2} \left( \frac{\partial^2 \psi(r, \theta, \phi)}{d\theta^2} + \cot(\theta) \frac{\partial \psi(r, \theta, \phi)}{\partial \theta, \phi} + \frac{1}{\sin^2(\theta)} \frac{\partial^2 \psi(r, \theta, \phi)}{d\phi^2} \right) \\
+ \frac{2\mu}{\hbar^2} (V(r) - E) \psi(r, \theta, \phi) = 0
\]

and its solutions

\[
\psi_{nlm}(r, \theta, \phi) = A_{nlm} \exp(-\rho^2/2) \ell^m_l(\rho) Y_{lm}(\theta, \phi)
\]

\[
\rho = \sqrt{\frac{\mu \omega}{\hbar}} r
\]
The 3-D Schrödinger Equation, with $V(r, \theta, \phi) = V(r) = -\frac{e^2}{4\pi\epsilon r}$

\[
\frac{\partial^2 \psi(r, \theta, \phi)}{dr^2} + \frac{2}{r} \frac{\partial \psi(r, \theta, \phi)}{\partial r} + \frac{1}{r^2} \left( \frac{\partial^2 \psi(r, \theta, \phi)}{d\theta^2} + \cot(\theta) \frac{\partial \psi(r, \theta, \phi)}{\partial \theta, \phi} + \frac{1}{\sin^2(\theta)} \frac{\partial^2 \psi(r, \theta, \phi)}{d\phi^2} \right) \\
+ \frac{2\mu}{\hbar^2} (V(r) - E) \psi(r, \theta, \phi) = 0
\]

and its solutions

\[
\psi_{nlm}(r, \theta, \phi) = A_{nlm} \rho^n \exp(-\rho^2/2) L_n(\rho) Y_{lm}(\theta, \phi)
\]

\[
\rho = \frac{me^2}{\hbar^2 4\pi\epsilon r}
\]
The solution of the 3-D Schrödinger Equation is

\[
|\psi(t)> = \sum_{nlm} \int dx dy dz \langle \psi_0 | x, y, z > \langle x, y, z | nlm > | nlm > \exp(-iE_{nlm} t/\hbar)
\]

which is equivalent to the matrix equation

\[
|\psi(t)> = \sum_{nlm} \langle \psi_0 | nlm > | nlm > \exp(-iE_{nlm} t/\hbar)
\]

\[
= \sum_{n_e} \langle \psi_0 | n_e > | n_e > \exp(-iE_{n_e} t/\hbar)
\]