Chapter 2
Principles of Steady-State Converter Analysis

2.1. Introduction

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2.6. Summary of key points

2.1. Introduction

Buck converter

SPDT switch changes dc component

Switch output voltage waveform

Duty cycle D:

0 ≤ D ≤ 1

complement D':

D' = 1 - D

Dc component of switch output voltage

Fourier analysis: Dc component = average value

\[
\langle v \rangle = \frac{1}{T} \int_0^T v(t) \, dt
\]

\[
\langle v \rangle = \langle DT_s V_g \rangle = DV_g
\]
Insertion of low-pass filter to remove switching harmonics and pass only dc component

\[ v \approx V_s = D V_g + L \frac{d(v)}{dt} \]

Objectives of this chapter

- Develop techniques for easily determining output voltage of an arbitrary converter circuit
- Derive the principles of inductor volt-second balance and capacitor charge (amp-second) balance
- Introduce the key small ripple approximation
- Develop simple methods for selecting filter element values
- Illustrate via examples
2.2. Inductor volt-second balance, capacitor charge balance, and the small ripple approximation

The small ripple approximation

In a well-designed converter, the output voltage ripple is small. Hence, the waveforms can be easily determined by ignoring the ripple:

\[ v(t) \approx V \]

\[ v(t) = V + v_{\text{ripple}}(t) \]

Buck converter analysis: inductor current waveform
Inductor voltage and current
Subinterval 1: switch in position 1

Inductor voltage
\[ v_L = V_g - v(t) \]
Small ripple approximation:
\[ v_L \approx V_g - V \]

Knowing the inductor voltage, we can now find the inductor current via
\[ v_L(t) = L \frac{di_L(t)}{dt} \]
Solve for the slope:
\[ \frac{di_L(t)}{dt} \approx \frac{v_L(t)}{L} \]
⇒ The inductor current changes with an essentially constant slope

Inductor voltage and current
Subinterval 2: switch in position 2

Inductor voltage
\[ v_L(t) = v(t) \]
Small ripple approximation:
\[ v_L \approx -V \]

Knowing the inductor voltage, we can again find the inductor current via
\[ v_L(t) = L \frac{di_L(t)}{dt} \]
Solve for the slope:
\[ \frac{di_L(t)}{dt} \approx \frac{-V}{L} \]
⇒ The inductor current changes with an essentially constant slope

Inductor voltage and current waveforms
Determination of inductor current ripple magnitude

\[
\Delta i_L = \frac{V_g - V_L}{2LDT_s} \Delta i_L
\]

Inductor current waveform during turn-on transient

When the converter operates in equilibrium:

\[
i_L((n+1)T_s) = i_L(nT_s)
\]

The principle of inductor volt-second balance:
Derivation

Inductor defining relation:

\[
v_L(t) = L \frac{di_L}{dt}
\]

Integrate over one complete switching period:

\[
i_L(T_s) - i_L(0) = \frac{1}{L} \int v_L(t) \, dt
\]

In periodic steady state, the net change in inductor current is zero:

\[
0 = \frac{1}{L} \int v_L(t) \, dt
\]

Hence, the total area (or volt-seconds) under the inductor voltage waveform is zero whenever the converter operates in steady state. An equivalent form:

\[
0 = \int v_L(t) \, dt = \int i_L \, dt
\]

The average inductor voltage is zero in steady state.
Inductor volt-second balance: Buck converter example

Inductor voltage waveform, previously derived:

Integral of voltage waveform is area of rectangles:
\[ \lambda = \int_{0}^{T_s} v_L(t) \, dt \]

Average voltage is
\[ \langle v_L \rangle = \frac{\lambda}{T_s} = D(V_g - V) + D'(V) \]

Equate to zero and solve for \( V \):
\[ 0 = DV_g - (D + D')V \quad \Rightarrow \quad V = DV_g \]

The principle of capacitor charge balance: Derivation

Capacitor defining relation:
\[ i_C(t) = C \frac{dv_C(t)}{dt} \]

Integrate over one complete switching period:
\[ \int_{0}^{T_s} i_C(t) \, dt = \int_{0}^{T_s} \frac{dv_C(t)}{dt} \, dt = v_C(T_s) - v_C(0) = \frac{C}{1} \frac{d}{dt} \int_{0}^{T_s} v_C(t) \, dt \]

In periodic steady state, the net change in capacitor voltage is zero:
\[ 0 = \frac{C}{1} \frac{d}{dt} \int_{0}^{T_s} v_C(t) \, dt \]

Hence, the total area (or charge) under the capacitor current waveform is zero whenever the converter operates in steady state. The average capacitor current is then zero.

2.3 Boost converter example
Boost converter analysis

Subinterval 1: switch in position 1

Inductor voltage and capacitor current
\[ v_L = V_g \]
\[ i_C = -\frac{v}{R} \]
Small ripple approximation:
\[ v_L = V_g \]
\[ i_C = -\frac{V}{R} \]

Subinterval 2: switch in position 2

Inductor voltage and capacitor current
\[ v_L = V_g - v \]
\[ i_C = -\frac{v}{R} \]
Small ripple approximation:
\[ v_L = V_g - V \]
\[ i_C = -\frac{I}{R} \]
Inductor voltage and capacitor current waveforms

\[ v_L(t) = V_g - V \cdot DT_s \]

\[ i_C(t) = \frac{V}{R} - \frac{V}{R} \cdot D_T_s \]

Inductor volt-second balance

Net volt-seconds applied to inductor over one switching period:

\[ \int_{0}^{T_s} v_L(t) \, dt = (V_g) \cdot DT_s + (V_g - V) \cdot D_T_s \]

Equate to zero and collect terms:

\[ V_g (D + D') - VD' = 0 \]

Solve for V:

\[ V = V_g \cdot D' \]

The voltage conversion ratio is therefore

\[ M(D) = \frac{V}{V_g} = \frac{1}{D'} = \frac{1}{1-D} \]
Determination of inductor current dc component

Capacitor charge balance:
\[
\int_{t_0}^{t} i_C(t) \, dt = -\frac{V}{R} DT_s + (I - \frac{V}{R}) DT_s.
\]
Collect terms and equate to zero:
\[
-\frac{V}{R} (D + D') + I = 0
\]
Solve for \( I \):
\[
I = \frac{V}{R} D'
\]
Eliminate \( I \) to express in terms of \( V_g \):
\[
i_C(t) = -\frac{V}{R} t DT_s - \frac{V}{R} D' DT_s
\]

Determination of inductor current ripple

Inductor current slope during subinterval 1:
\[
\frac{dI}{dt}(t_1) = \frac{V}{R} - V_L
\]
Inductor current slope during subinterval 2:
\[
\frac{dI}{dt}(t_2) = \frac{V}{R} - V_L DT_s
\]
Change in inductor current during subinterval 1 is (slope) (length of subinterval):
\[
\Delta I_1 = \frac{V}{R} DT_s
\]
Solve for peak ripple:
\[
\Delta I = \frac{V}{R} DT_s
\]
Choose \( L \) such that desired ripple magnitude is obtained.

Determination of capacitor voltage ripple

Capacitor voltage slope during subinterval 1:
\[
\frac{dV}{dt}(t_1) = \frac{I}{C} - \frac{V}{R C}
\]
Capacitor voltage slope during subinterval 2:
\[
\frac{dV}{dt}(t_2) = \frac{I}{C} - \frac{V}{R C}
\]
Change in capacitor voltage during subinterval 1 is (slope) (length of subinterval):
\[
\Delta V_1 = \frac{V}{R C} DT_s
\]
Solve for peak ripple:
\[
\Delta V = \frac{V}{R C} DT_s
\]
Choose \( C \) such that desired voltage ripple magnitude is obtained.
In practice, capacitor equivalent series resistance (ESR) leads to increased voltage ripple.
2.4 Cuk converter example

Cuk converter, with ideal switch:

\[ \begin{align*}
L_1 &+ C_1 + R + v_2 \quad C_2 \quad L_2 \\
&+ v_1 - i_2 \quad v_1 \\
\text{Switch in position 1:} &
\text{MOSFET conducts} \\
\text{Capacitor } C_1 \text{ releases} &
\text{energy to output} \\
\text{Switch in position 2:} &
\text{diode conducts} \\
\text{Capacitor } C_1 \text{ is} &
\text{charged from input}
\end{align*} \]

Cuk converter: practical realization using MOSFET and diode:

\[ \begin{align*}
L_1 &+ C_1 + R + v_2 \quad C_2 \quad L_2 \\
&+ v_1 - i_2 \quad v_1 \\
\text{Switch in position 1:} &
\text{MOSFET conducts} \\
\text{Capacitor } C_1 \text{ releases} &
\text{energy to output} \\
\text{Switch in position 2:} &
\text{diode conducts} \\
\text{Capacitor } C_1 \text{ is} &
\text{charged from input}
\end{align*} \]

Waveforms during subinterval 1

MOSFET conduction interval

Inductor voltages and capacitor currents:

\[ \begin{align*}
v_{L1} &= V_s \\
v_{L2} &= -v_1 - v_2 \\
i_C1 &= i_2 \\
i_C2 &= i_2 - \frac{v_2}{R}
\end{align*} \]

Small ripple approximation for subinterval 1:

\[ \begin{align*}
v_{L1} &= V_s \\
v_{L2} &= -v_1 - v_2 \\
i_C1 &= i_2 \\
i_C2 &= i_2 - \frac{v_2}{R}
\end{align*} \]
Waveforms during subinterval 2
Diode conduction interval

Inductor voltages and capacitor currents:

\[ v_L = V_2 - V_1 \]
\[ v_C = -v_i \]
\[ i_L = i_i \]
\[ i_C = \frac{i_2}{2} \]

Small ripple approximation for subinterval 2:

\[ v_L = V_2 - V_1 \]
\[ v_C = -v_i \]
\[ i_L = i_i \]
\[ i_C = \frac{i_2}{2} \]

Equate average values to zero

The principles of inductor volt-second and capacitor charge balance state that the average values of the periodic inductor voltage and capacitor current waveforms are zero, when the converter operates in steady state. Hence, to determine the steady-state conditions in the converter, let us sketch the inductor voltage and capacitor current waveforms, and equate their average values to zero.

Waveforms:

Inductor voltage \( v_L (t) \):

\[ v_L(t) = D V_g + D' (V_2 - V_1) = 0 \]

Volt-second balance on \( L_1 \):

\[ \int_{t_0}^{t_1} v_L(t) \, dt = DV_1 + D' (V_2 - V_1) = 0 \]

Capacitor \( C \) current:

\[ i_C(t) = \frac{1}{C} \int_{t_0}^{t_1} v_C(t) \, dt \]

\[ i_C(t) = DI_2 + D' I_1 = 0 \]

Equate average values to zero

Inductor \( L \) voltage:

\[ v_L(t) = -V_2 \]

Average the waveforms:

\[ \frac{1}{D} \int_{t_0}^{t_1} v_L(t) \, dt = -V_1 \]

Capacitor \( C \) current:

\[ i_C(t) = I_2 \]

Average the waveforms:

\[ \frac{1}{D} \int_{t_0}^{t_1} i_C(t) \, dt = I_1 \]
Equate average values to zero

Capacitor current \( i_C(t) \) waveform

\[
\frac{1}{D_T} \int_{D_T}^{(1-D_T)} (i_C) = I_2 - \frac{V_2}{R} > 0
\]

Note: during both subintervals, the capacitor current \( i_C \) is equal to the difference between the inductor current \( i_2 \) and the load current \( V_2/R \). When ripple is neglected, \( i_C \) is constant and equal to zero.

Cuk converter conversion ratio \( M = V/V_g \)

\[
M(D) = \frac{V}{V_g} = -\frac{D}{1-D}
\]

Inductor current waveforms

Interval 1 slopes, using small ripple approximation:

\[
\frac{di_1(t)}{dt} = \frac{V_g - V_1}{L_1} = 1
\]

Interval 2 slopes:

\[
\frac{di_2(t)}{dt} = \frac{V_1 - V_2}{L_2} = -1
\]
Capacitor $C_1$ waveform

Subinterval 1:
$$\frac{dv_1(t)}{dt} = \frac{i_{C_1}(t)}{C_1}$$
Subinterval 2:
$$\frac{dv_1(t)}{dt} = \frac{i_{C_1}(t)}{C_1}$$

Ripple magnitudes

Analysis results
- $\Delta i_1 = \frac{V_g D T_s}{2 L_1}$
- $\Delta i_2 = \frac{V_1 + V_2}{2 L_2} D T_s$
- $\Delta v_1 = -\frac{I_2}{2 C_1} D T_s$

Use dc converter solution to simplify:
- $\Delta i_1 = \frac{V_g D T_s}{2 L_1}$
- $\Delta i_2 = \frac{V_g D T_s}{2 L_2}$
- $\Delta v_1 = \frac{V_g D}{2 R C_1}$

Q: How large is the output voltage ripple?

2.5 Estimating ripple in converters containing two-pole low-pass filters

Buck converter example: Determine output voltage ripple
Capacitor current and voltage, buck example

Must not neglect inductor current ripple!

If the capacitor voltage ripple is small, then essentially all of the ac component of inductor current flows through the capacitor.

$i_C(t)$
$v_C(t)$

Estimating capacitor voltage ripple $\Delta v$

Current $i_C(t)$ is positive for half of the switching period. This positive current causes the capacitor voltage $v_C(t)$ to increase between its minimum and maximum extrema. During this time, the total charge $q$ is deposited on the capacitor plates, where

\[ q = C (\Delta v) \]

\[ (\text{change in charge}) = C (\text{change in voltage}) \]

The total charge $q$ is the area of the triangle, as shown:

\[ q = \frac{1}{2} \Delta i_L T \]

Eliminate $q$ and solve for $\Delta v$:

\[ \Delta v = \frac{\Delta i_L}{C} \]

Note: In practice, capacitor equivalent series resistance (esr) further increases $\Delta v$. 

Fundamentals of Power Electronics

Chapter 2: Principles of steady-state converter analysis
Inductor current ripple in two-pole filters

Example: problem 2.9

\[ \lambda = L \Delta i \]

\[ \lambda = \text{inductor flux linkages} \]

\[ \Delta t = \text{inductor volt-seconds} \]

2.6 Summary of Key Points

1. The dc component of a converter waveform is given by its average value, or the integral over one switching period, divided by the switching period. Solution of a dc-dc converter to find its dc, or steady-state, voltages and currents therefore involves averaging the waveforms.

2. The linear ripple approximation greatly simplifies the analysis. In a well-designed converter, the switching ripples in the inductor currents and capacitor voltages are small compared to the respective dc components, and can be neglected.

3. The principle of inductor volt-second balance allows determination of the dc voltage components in any switching converter. In steady-state, the average voltage applied to an inductor must be zero.

4. The principle of capacitor charge balance allows determination of the dc components of the inductor currents in a switching converter. In steady-state, the average current applied to a capacitor must be zero.

5. By knowledge of the slopes of the inductor current and capacitor voltage waveforms, the ac switching ripple magnitudes may be computed. Inductance and capacitance values can then be chosen to obtain desired ripple magnitudes.

6. In converters containing multiple-pole filters, continuous (nonpulsating) voltages and currents are applied to one or more of the inductors or capacitors. Computation of the ac switching ripple in these elements can be done using capacitor charge and/or inductor flux-linkage arguments, without use of the small-ripple approximation.

7. Converters capable of increasing (boost), decreasing (buck), and inverting the voltage polarity (buck-boost and Cuk) have been described. Converter circuits are explored more fully in a later chapter.

Summary of Chapter 2

- The dc component of a converter waveform is given by its average value, or the integral over one switching period, divided by the switching period. Solution of a dc-dc converter to find its dc, or steady-state, voltages and currents therefore involves averaging the waveforms.

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- The principle of inductor volt-second balance allows determination of the dc voltage components in any switching converter. In steady-state, the average voltage applied to an inductor must be zero.

- The principle of capacitor charge balance allows determination of the dc components of the inductor currents in a switching converter. In steady-state, the average current applied to a capacitor must be zero.

- By knowledge of the slopes of the inductor current and capacitor voltage waveforms, the ac switching ripple magnitudes may be computed. Inductance and capacitance values can then be chosen to obtain desired ripple magnitudes.

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