Middlebrook's Feedback Theorem

The simple block diagram developed previously is useful only if the feedback circuit does not load the output or input, and if the various gains which comprise $T$ and $R$ can be easily identified (i.e., no interaction between blocks due to loading), and if there is no direct forward transmission through the feedback path (i.e., blocks are truly unidirectional).

Middlebrook's Feedback theorem is a more general technique which allows more complex feedback systems, containing interaction between blocks, to be analyzed almost by inspection. "Loop gain" and "Amount of feedback" are emphasized. The theorem can be thought of as a generalization of the loop gain measurement technique described previously.
Method and Proof  (due to Middlebrook)

Find a point inside the loop where a controlled current generator output is proportional to the error signal, and can be diverted to ground so that no amplified error signal is conveyed to the output:

A transistor collector terminal (class A linear amplifier) is a point which typically satisfies the above requirements. Insert an independent current source \( i_2 \) at this point.
The system now has two independent inputs — $v_i$ and $i_x$. The output $v_o$ contains components arising from both of these inputs. We will use superposition to compute $v_o$.

The theorem can be developed in dual form, in which an independent voltage generator $v_E$ is inserted in series with a controlled voltage generator whose output is proportional to the error signal:
In either case, be careful to follow the above rules in selecting a suitable injection point. To test the validity of an injection point, consider breaking the feedback connection (k→e) Then, $i_g$ (or $v_y$) should be a function of $v_e$, and should be independent of $i_x$ and $i_2$ (or $v_x$ and $v_2$). Also, if $i_x$ (or $v_x$) is zero, then the output $v_o$ should be zero, regardless of $v_e$.

Note that the first test is not satisfied when there is an impedance between the generator $y_{uv,e}$ (or $h_{uv,e}$) and the injection point.

Now, $i_g = y_{uv,e} + i_x \frac{z_2}{z_1} = \frac{z_1}{z_1 + z_2} y_{uv,e} + \frac{z_2}{z_1 + z_2} i_2$

which is not independent of $i_2$ or $i_x$.

The second test is not satisfied when injection is attempted in one of multiple parallel paths inside a feedback loop.
Computation of $\nu_0$ using superposition:

1. Original condition: $i_2 = 0$

Then the closed-loop forward gain $G$ is given by:

$$\nu_0 \bigg|_{i_2=0} = G \nu_i \quad \text{(definition of } G) \quad (1)$$

Also, we can express $i_x$ and $i_y$ in terms of $\nu_i$:

$$i_x \bigg|_{i_2=0} = -i_y \bigg|_{i_2=0} = G_a \nu_i \quad \text{(definition of } G_a) \quad (2)$$

2. Set $\nu_i = 0$, and inject $i_2$.

![Diagram]

Then

$$i_y \bigg|_{\nu_i=0} = T i_x \bigg|_{\nu_i=0} \quad \text{(definition of } T, \text{ the "loop gain")} \quad (3)$$

In addition, $i_x$ and $i_y$ can be expressed in terms of $i_2$:

$$i_x + i_y = i_2 \quad \text{(4)}$$
\[ \Rightarrow \quad i_x \Big|_{v_t=0} + T i_x' \big|_{v_t=0} = i_2 \quad \text{(5)} \]

or, \((1+T) \ i_x \Big|_{v_t=0} = i_2 \quad \text{(6)}\]

Hence,
\[ i_x \big|_{v_t=0} = \frac{1}{1+T} \ i_2 \quad \text{(7)} \]
\[ i_y \big|_{v_t=0} = \frac{T}{1+T} \ i_2 \quad \text{(8)} \]

Also, define
\[ v_0 \big|_{v_t=0} \triangleq G_b \ i_x \big|_{v_t=0} = \frac{G_b}{1+T} \ i_2 \quad \text{(9)} \]

(definition of \( G_b \))

3. In Presence of Both \( v_t \) and \( i_2 \)

Use superposition:
\[ i_x = i_x \big|_{v_t=0} + i_x \big|_{v_t=0} \quad \text{(10)} \]

Insert Eqs. (2) and (7):
\[ \Rightarrow \quad i_x = G_a v_t + \frac{1}{1+T} \ i_2 \quad \text{(11)} \]

Similar for \( i_y \)
\[ i_y = i_y \big|_{v_t=0} + i_y \big|_{v_t=0} \quad \text{(12)} \]
Insert Eqs. (2) and (8):

\[ i_y = -G_a v_i + \frac{T}{1+T} i_x \]  

(13)

Determination of output via superposition:

\[ v_o = v_o \bigg|_{v_i=0} + v_o \bigg|_{i_x=0} \]  

(14)

Substitute using Eqs. (9) and (1):

\[ v_o = G v_i + \frac{G_b}{1+T} i_x \]  

(15)

Now let us consider performing a thought experiment in which \( i_x \) and \( v_i \) are adjusted such that some special conditions are satisfied.

1. Adjust \( i_x \) (\( i_x = 0 \)) as necessary such that \( i_x \) is null, i.e., \( i_x = 0 \), in presence of \( v_i \).

\[ 0 = G_a v_i + \frac{1}{1+T} i_x \bigg|_{i_x=0} \]  

(From Eq. (11))  

(16)

Also, from Eq. (15),

\[ v_o \bigg|_{i_x=0} = G v_i + \frac{G_b}{1+T} i_x \bigg|_{i_x=0} \]  

(17)
Eliminate $i_2 |_{i_x=0}$:

$$v_0 |_{i_x=0} = G v_i - G_a G_b v_i$$  \hspace{1cm} (18)

Define $G_0 = G - G_a G_b$

so $v_0 |_{i_x=0} = G_0 v_i$  \hspace{1cm} (20)

**Physical interpretation**

In this thought experiment, we are adjusting $i_2$ as necessary such that there is no transmission through the forward path, i.e., $i_x = 0$. The only other way to obtain a nonzero output is via the feedback path, through the nonideal block $k$. Hence,

$$G_0 = \text{Direct forward transmission through feedback path}$$
(ii) Adjust \( i_2 \) (\( \rightarrow i_2|_{i_y=0} \)) as necessary such that \( i_y \) is nulled, i.e., \( i_y=0 \), in presence of \( v_i \):

\[
\Rightarrow \quad 0 = -G_a v_i + \frac{1}{1+T} i_2|_{i_y=0}
\]

(From Eq. (13))

Also from Eq. (15),

\[
v_o|_{i_y=0} = G v_i + \frac{G_a G_b}{1+T} i_2|_{i_y=0}
\]

Eliminate \( i_2|_{i_y=0} \):

\[
v_o|_{i_y=0} = G v_i + \frac{G_a G_b}{T} v_i
\]

Define \( G_{\infty} = G + \frac{G_a G_b}{T} \)

so \( v_o|_{i_y=0} = G_{\infty} v_i \)

**Physical Interpretation**

In thought experiment (ii), we are adjusting \( i_2 \) as necessary such that \( i_y=0 \), and hence the error signal is nulled. Hence, \( G_{\infty} \) is the ideal closed-loop gain (\( = \frac{v_o}{v_i} \)) when
error is zero). Thought experiment (ii) amounts to the "virtual short principle" commonly used in op-amp circuit analysis.

(iii) Adjust $i_2 \ (\rightarrow i_2 \bigg|_{v_o=0})$ as necessary such that the output $v_o$ is nulled to zero, in presence of $v_i$. Note that the output is not shorted.

From Eq. (15),

$$0 = Gv_i + \frac{G_p}{1+T} i_2 \bigg|_{v_o=0}$$  \hspace{1cm} (26)

From Eq. (11),

$$i_k \bigg|_{v_o=0} = G_a v_i + \frac{1}{1+T} i_2 \bigg|_{v_o=0}$$  \hspace{1cm} (27)

From Eq. (13),

$$i_y \bigg|_{v_o=0} = -G_a v_i + \frac{T}{1+T} i_2 \bigg|_{v_o=0}$$  \hspace{1cm} (28)
Now eliminate \( v_i \) and \( i_2 \) \( \left| v_o = 0 \right. \):

\[
y \left| v_o = 0 \right. = \frac{TG + GaGb}{G - GaGb} \quad i \left| v_o = 0 \right.
\]

(29)

Define \( T_n = \frac{iy}{ik} \left| v_o = 0 \right. = \frac{TG + GaGb}{G - GaGb} \)

\( T_n = \) "null loop gain" \( \frac{iy}{ik} \) in presence of \( v_i \), when \( i_2 \) is adjusted to null the output.

Using the above definitions, we can now solve the circuit for \( G_0 \), \( G_{oo} \), \( T_n \), and \( T \). We can then determine \( G \) as follows (note that the intermediate quantities \( G_a \) and \( G_b \) are also unknown at this point):

We have

\[
G_0 = G - GaGb \quad \text{from Eq. (19)}
\]

\[
G_{oo} = \frac{G + GaGb}{T} \quad \text{from Eq. (24)}
\]

Hence

\[
G_{oo} T = TG + GaGb \quad \text{(31)}
\]

and

\[
T_n = \frac{TG + GaGb}{G - GaGb} \quad \text{from Eq. (30)}
\]

Substitute Eqs. (19) and (31) into Eq. (30):

\[
T_n = \frac{G_{oo} T}{G_0} \quad \text{or} \quad \frac{T_n}{T} = \frac{G_{oo}}{G_0} \quad \text{(32)}
\]
So we only need to solve for three of the gains; the fourth is then given by Eq. (32).

Now eliminate \( G_a G_b \), and solve for \( G \).

The result is:

\[
G = G_0 \cdot \frac{1 + \frac{1}{T_a}}{1 + \frac{1}{T}} = G_0 \cdot \frac{T}{1 + T} + G_o \cdot \frac{1}{1 + T}
\] (33)

This is the desired expression for the closed-loop gain \( G \). Note that

\[
G \to G_0 \quad \text{as} \quad \|T\| \to \infty
\]

\[
G \to G_o \quad \text{as} \quad \|T\| \to 0
\]
### Summary

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Current Injection $i_e$</th>
<th>Voltage Injection $v_e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T(s)$</td>
<td>$T(s) = \frac{i_y}{i_k} \bigg</td>
<td>\quad i_k = 0$</td>
</tr>
<tr>
<td>$T_u(s)$</td>
<td>$T_u(s) = \frac{i_y}{i_k} \bigg</td>
<td>\quad u_o = 0$</td>
</tr>
<tr>
<td>$G_{oo}(s)$</td>
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<td>\quad i_k = 0$</td>
</tr>
</tbody>
</table>

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$u_o$ and $u_i$ are the output and input signals, and may be voltages, currents or whatever.

with

$$\frac{u_o}{u_i} = G = \frac{T}{1+T} \quad G_{oo} + \frac{1}{1+T} \quad G_o$$