Some Applications

1. Analysis of an adaptive pole placement system

Our plant

\[
\begin{align*}
  \frac{d}{s + \beta} & : Y_p \\
  \quad & \quad \\
  U_p & \rightarrow Y_p
\end{align*}
\]

We only know \( d \) is positive (don't know \( \beta \) exact a value).
We'd like a controller which gives closed-loop response.

One approach:

\[
\begin{align*}
  \frac{1}{s + 1} & : Y_m \\
  \quad & \quad \\
  r & \rightarrow Y_m
\end{align*}
\]

\[
\begin{align*}
  \frac{1}{s + 1} & : Y_m \\
  \quad & \quad \\
  e & = Y_p - Y_m
\end{align*}
\]

\[
\begin{align*}
  \frac{d}{s + \beta} & : Y_p \\
  \quad & \quad \\
  Y_m & \rightarrow Y_p
\end{align*}
\]

We'll adapt these gains to make \( e \) small.
Clearly, if we knew \( \beta \) we'd like

\[
c = \frac{1}{\alpha} c^* \\
d = \frac{\beta - 1}{\alpha} d^*
\]

which would give \( \epsilon \equiv 0 \). But of course, we don't know \( \alpha \) or \( \beta \).

One scheme adjusts \( c \) and \( d \) as follows:

\[
\begin{align*}
c &= -\gamma r e \\
d &= -\gamma y p e
\end{align*}
\]

\( \gamma > 0 \) is called the "adaptation gain" — we pick it.

Now have

\[
\begin{bmatrix}
Y_p \\
Y_m \\
c \\
d
\end{bmatrix} =
\begin{bmatrix}
-\beta Y_p + \alpha (c r + d y p) \\
-\gamma m + r \\
-\gamma r (Y_p - Y_m) \\
-\gamma y p (Y_p - Y_m)
\end{bmatrix}
\]

Define

\[
\phi =
\begin{bmatrix}
c - c^* \\
d - d^*
\end{bmatrix} = \text{"gain errors"}
\]

\[
W =
\begin{bmatrix}
r \\
Y_p
\end{bmatrix} = \text{"regressor vector"}
\]
Redraw as:

\[
\begin{align*}
\dot{e} &= -e + \alpha \phi^T \mathbf{w} \\
\dot{\phi} &= -\gamma e \mathbf{w}
\end{align*}
\]

\[\text{error equations}\]

Let \( V(e, \phi) : \mathbb{R}^3 \rightarrow \mathbb{R}_+ \) which is clearly PD in \( \mathbb{R}^3 \).

\[
DV = \begin{bmatrix}
\frac{\partial V}{\partial e} \\
\frac{\partial V}{\partial \phi}
\end{bmatrix} = \begin{bmatrix}
\gamma e \\
\alpha \phi^T
\end{bmatrix}
\]

\[
DV(e, \phi) f(e, \phi) = -\gamma e^2 + \gamma \alpha e \phi^T \mathbf{w} - \gamma \alpha e \phi^T \mathbf{w} = -\gamma e^2 \leq 0 \quad \forall e, \phi
\]

\[
\therefore e = 0, \phi = [0] \text{ is a globally stable equilibrium point.}
\]

In particular \( e \) and \( \phi \) are bounded.
Then if r is bounded so is ym (why?) and we quickly get all signals bounded.

In fact, (see discussion on page 108) sublevel sets of V(e, ϕ) are forward invariant and

\[ V(e(t), \phi(t)) \leq V(e(0), \phi(0)) \]

Can we show asymptotic stability? La Salle says look where \( V = 0 \)

\[-ye^2 = 0 \iff e = 0\]

but, unfortunately, \( e(t) = 0 \) \( \phi(t) = \phi_0 \) where \( \phi_0^TW = 0 \)

are non-zero solutions to the error equations which lie totally inside the set \( \{ x | V(x) = 0 \} \). La Salle's Theorem is of no help;

But from the proof of LaSalle's Theorem we get that \( e(t) \to 0 \) (since the proof showed \( W(z) \leq \{ x | V(x) = 0 \} \) and we know by Lemma on page 110 that \( e(t) \to w(z) \neq 2 \))

In other words the plant tracks asymptotically the model V. (Later, we'll see another way to show \( e(t) \to 0 \).

Note: In general \( \phi(t) \to 0 \) unless you put conditions on r. This is the subject of parameter convergence in adaptive control, a tricky thing.
2. Robot feedback control

\[ \dot{\theta} = w \]
\[ \dot{w} = f(\theta, w, \gamma) \]

\( \text{has a ton of sines & cosines in it very ugly loosely.} \)

Let \( E(\theta, w) \) denote kinetic energy of robot, so if \( \theta, w \)
solve the DE above

\[ \frac{d}{dt} E(\theta(t), w(t)) \leq W^T \gamma \]

(power done on robot)

(difference is dissipation, if any)

Note that:

\[ E(\theta, w) \geq 0 \]
\[ E(\theta, w) = 0 \Rightarrow w = 0 \]
\[ E(\theta, w) \to \infty \text{ as } w \to \infty \]
Here's a position controller

\[ \gamma = -\text{SAT}(\theta - \theta_{\text{ref}}) - \text{SAT}(w) \]

\[ \text{SAT} : \mathbb{R}^m \rightarrow \mathbb{R}^m \]

is given by

\[ (\text{SAT}(x))_i = \text{Sat} \left( x_i \right) \]

(is componentwise sat operation)

where

\[ \text{Sat}(x) = \begin{cases} x & |x| < 1 \\ \text{sgn}(x) & |x| \geq 1 \end{cases} \]
Let us now establish that the controller works, i.e., $[\Theta_{\text{ref}}, 0]^T$ is a globally asymptotically stable equilibrium point of the closed loop.

Closed-loop equations are:

$$\dot{\Theta} = w$$
$$w = f(\Theta, w, -\text{SAT}(\Theta-\Theta_{\text{ref}}) - \text{SAT}(\omega))$$

Define

$$\Psi(x) = \int_0^x \text{sat}(u) du = \begin{cases} \frac{x^2}{2} & 1 \leq 1 \\ 1x - \frac{1}{2} & 1 \leq 1 \end{cases}$$

Here's a Lyapunov function:

$$V(\Theta, w) = E(\Theta, w) + \Psi(\Theta-\Theta_{\text{ref}})$$

where $\Psi(\Theta) = \sum_i \Psi(\Theta_i)$ (i.e., $\Psi$ is $\Psi$ above applied component-wise to vectors and summed.)

Check that $V(\Theta, w)$ is PD (around $[\Theta_{\text{ref}}, 0]$).

$$DV = \left[ \frac{\partial E}{\partial \Theta} + \text{SAT}(\Theta-\Theta_{\text{ref}})^T, \frac{\partial E}{\partial w} \right]$$

so

$$\dot{V} = \frac{\partial E}{\partial \Theta} w + \frac{\partial E}{\partial w} f(\Theta, w, \text{SAT}(\Theta-\Theta_{\text{ref}}) - \text{SAT}(\omega)) + \text{SAT}(\Theta-\Theta_{\text{ref}})^T w$$

$$= \frac{d}{dt} E(\Theta(t), w(t)) + \text{SAT}(\Theta-\Theta_{\text{ref}})^T w$$

$$\leq \underbrace{w^T \gamma}_{\text{robot power input}} + \text{SAT}(\Theta-\Theta_{\text{ref}})^T w$$
\[ W^T(-\text{SAT}(\theta-\theta_{\text{ref}}) - \text{SAT}(w)) + \text{SAT}(\theta-\theta_{\text{ref}})^TW \]

\[ = -W^T\text{SAT}(w) \leq 0 \]

By Lyapunov's Theorem we conclude the system is globally stable. We do not have asymptotic stability (yet) since \[ -\dot{V} = W^T\text{SAT}(w) \] is not PD (around \[ \theta_{\text{ref}} \]).

But LaSalle's Theorem saves us.

\[ \dot{V} = 0 \iff W = 0 \]

and \[ W = 0 \implies \theta \text{ is constant} \implies \dot{W} = 0. \] So, what solutions \[ \theta \] are there to:

\[ 0 = f(\theta, 0, -\text{SAT}(\theta-\theta_{\text{ref}})) \]

Answer: Only \( \theta = \theta_{\text{ref}} \) for our robot.

Thus LaSalle's theorem gives global asymptotic stability (about \( [\theta_{\text{ref}}, 0] \)). Our controller works - i.e. it takes the robot to its commanded reference at rest.
A Look at Discrete-Time Systems

For \( x_{k+1} = f(x_k, K) \)
the equilibrium points are those \( xe \) such that
\[ x_e = f(x_e, K) \]
I.E. they are fixed points of the map \( f \).

We can pull our usual trick of translating any given
equilibrium point to the origin since if we let
\[ \tilde{x}_k = x_k - x_e \]
then \( x_k \) is a solution to \( x_{k+1} = f(x_k, K) \)
iff \( \tilde{x}_k \) is a solution to
\[ \tilde{x}_{k+1} = \tilde{f}(\tilde{x}_k, K) \]
where \( \tilde{f}(\tilde{x}_k, K) = f(\tilde{x}_k + x_e, K) - x_e \)

Note \( \tilde{f}(0, K) = 0 \)

All of the definitions for stability, etc. given on
page 54 carry over to discrete-time if we define the
discrete-time versions of \( C^\infty \) and \( C^0 \)
\[ C^\infty(Z^+) = \{ \phi: Z^+ \to \mathbb{R}^n \mid \|\phi(K)\|_2 \text{ is bounded} \} \]
where \( Z^+ = \) non-negative integers = \( \{0, 1, 2, \ldots\} \)

We leave it to the reader to define \( C^0(Z^+) \) for himself.
The process of "translating" to discrete-time is fairly
painless usually involving a simple substitution \( Z \to \mathbb{R} \) or
\( Z \to \mathbb{R}^+ \) of the underlying "time set."
Notice that in the definition of the discrete-time version of $C^r$ that the requirement "$\phi$ is continuous" was not needed. Basically, any map of the integers ($\mathbb{Z}$) is automatically continuous because $\mathbb{Z}$ is a discrete space.

If you take this delightful situation as a foreshadow, it may be expected that many things turn out nicely in discrete time. For the most part, this is true. E.g. we only need $f(x)$ \underline{continuous} on $\mathbb{R}$ to guarantee uniqueness and existence of solutions to

$$x_{k+1} = f(x_k)$$

No Lipschitz conditions are required.

The next major question might be: How does discrete-time Lyapunov Theory go? The answer is fairly surprising.

There are some substantial differences and in discrete time one must be careful, especially concerning local stability results. Care is needed because a D.T. system can "leap" from

$$x_k \in \{x : V(x) \leq K\} \text{ to } x_{k+1} \in \{x : V(x) > K\}$$

without ever "touching" the set $\{x : V(x) = K\}$

Let's state and prove a local D.T. stability result. First, we need a definition.
DEF: A family of functions, $\phi_K : (V_a, \| \cdot \|_a) \to (V_b, \| \cdot \|_b)$ mapping a vector-space $V_a$ (with norm $\| \cdot \|_a$) into a vector space $V_b$ (with norm $\| \cdot \|_b$) is said to be EQUICONTINUOUS at a point $x_0 \in V_a$ iff given $\varepsilon > 0 \exists \delta > 0 \exists$

\[ \| x_0 - x \|_a < \delta \implies \| \phi_K(x_0) - \phi_K(x) \|_b < \varepsilon \quad \forall K \]

Now our theorem:

THM: Suppose $f(x,K)$ is such that the family of functions

\[ h_K(x) \triangleq f(x,K) \]

is equicontinuous at $x=0$. If there is a $V: U \subseteq \mathbb{R}^n \to \mathbb{R}$ which is continuous and LPO (near 0) and there is an $r > 0 \exists \ x \in B_r \implies \Delta V(x,K) \triangleq V(f(x,K)) - V(x) \leq 0 \ \forall K \geq 0$

Then $x_{K+1} = f(x_K,K)$ is stable (at 0).

If in addition $\inf_K - \Delta V(x,K)$ is LPO (near 0)

Then $x_{K+1} = f(x_K,K)$ is asymptotically stable.

Proof: As in the proof of the Cont. -Time theorem on page 66, find $r > 0 \exists$

\[ x \in B_r \implies \]

\[ i) \ V(x) \geq 0 \]
\[ ii) \ V(x) = 0 \iff x = 0 \]
\[ iii) \ \Delta V(x,K) \leq 0 \ \forall K \geq 0 \]
Let $\epsilon > 0$ be given and define $\epsilon^* = \min(\epsilon, r)$

Use the equicontinuity of $f(x,y,k)$ to find $R > 0$ so that

$$\|x\|_\infty < R \Rightarrow \|f(x,y,k)\|_\infty < \epsilon$$

and define $R^* = \min(\epsilon, R)$.

Let

$$K = \inf \left\{ V(z) \mid \min(\epsilon^*, R^*) \leq \|z\|_\infty \leq r \right\}$$

As usual, a key property of $K$ is that

1) $V(x) < K$ & $\|x\|_\infty < R^* \Rightarrow \|x\|_\infty < \epsilon^*$

2) $V(x) < K$ & $\|x\|_\infty < r \Rightarrow \|x\|_\infty < \epsilon^*$

By continuity of $V$, find $\delta > 0$ so that

$$\|x\|_\infty < \delta \Rightarrow V(x) < K \quad \text{and let } \delta^* = \min(\delta, \epsilon^*)$$

I claim that $\|x\|_\infty < \delta^* \Rightarrow \|\varphi(x_0)(k)\|_\infty < \epsilon^* \quad \forall k$

If I can show my claim, the proof is done...
Let $x_k$ be a solution to $x_{k+1} = f(x_k, k)$ and $\|x_0\| < \delta$. To simplify notation let

$$V_k := V(x_k).$$

First note that if $k$ is such that $\|x_k\| \leq R^*$ \(i = 0, 1, 2, \ldots, k\) then $V_k \leq V_{k-1} \leq V_{k-2} \ldots \leq V_0 < K$

I will first show $\|x_k\| < R^*$ \(\forall k\). Suppose not, and let $k^*$ be the first index where $\|x_k\| \geq R^*$. Now, evidently then $\|x_{k^* - 1}\| \leq R^* \leq R$ and (since $R^* \leq R$)

$$V_{k^* - 1} \leq V_0 < K$$

By property i) of $k$ (on page 146) we conclude either a) $\|x_{k^* - 1}\| < \delta$ or b) $\|x_{k^* - 1}\| \geq R^*$. But a) is impossible since it implies $\|x_{k^* - 1}\| < R^* < R$, which is impossible since $k^*$ was first exit.

With $\|x_k\| < R^*$ \(\forall k\) thus established, I will now show $\|x_k\| \leq \varepsilon^* < \varepsilon$. If $\|x_k\| \geq \varepsilon^*$

we get from $V_k \leq V_0 < K$ and property iv) of $k$

that $\|x_k\| > R$ which is impossible.

Thus no solution with $\|x_0\| < \delta$ can have $\|x_k\| \geq \varepsilon^*$ and stability is proven. For asymptotic stability, choose $\varepsilon$ within $\Delta V$'s LPO domain and pick $\delta$ by the stability proof. Then $\|x_0\| < \delta \Rightarrow \|x_k\| \to 0$

You supply details in HW.