Quick review on norms.

\( x : \text{vector in } \mathbb{R}^n \quad \|x\|_p = (|x_1|^p + \ldots + |x_n|^p)^{1/p} \)

- \( p : \text{Euclidean norm} \)
- \( p = \infty \quad \|x\|_\infty = \max_{1 \leq i \leq n} |x_i| \)
- \( p = 0 \quad \|x\|_0 = \text{card} \{ i : |x_i| \neq 0 \} \)

\[ |x^T y| \leq \|x\|_p \|y\|_q \quad \frac{1}{p} + \frac{1}{q} = 1 \quad \text{Holder inequality} \]

Matrix norm

- \( \|A\|_p = \sup_{\|x\|_p = 1} \|Ax\|_p = \sup_{\|x\|_p = 1} \|A\|_p \)

- Frobenius: \( \|A\|_F = \left( \sum_{i,j} |a_{ij}|^2 \right)^{1/2} \)

Properties

\[ \|A\|_1 = \max_{j} \sum_{i} |a_{ij}| \]
\[ \|A\|_\infty = \max_{i} \sum_{j} |a_{ij}| \]
In the rest of the course we may only work in a metric space. Let $X$ be a set of points (e.g., nodes on a graph). Define a distance function $d : X \times X \to \mathbb{R}_+$ with the properties:

- $d(i, j) = d(j, i)$
- $d(i, i) = 0$ if and only if $i = j$
- $d(i, j) \leq d(i, k) + d(k, j)$

**Example:**

Let $x = (x_1, \ldots, x_n)$ in the $\ell_p$ (Minkowski) norm

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

- For $p = 1$, $\|x\|_1 = \sum |x_i|$ is the Manhattan norm.
- For $p = 2$, $\|x\|_2 = \sqrt{\sum x_i^2}$ is the Euclidean norm.
- For $p = \infty$, $\|x\|_\infty = \max |x_i|$ is the Chebyshev norm.

The unit ball is defined as $x \in \mathbb{R}^n$ such that $\|x\|_p \leq 1$. The shape of the unit ball changes with different values of $p$. A diagram illustrates these changes.
Unit cube: \( [0,1]^n \) : \( \mathbb{Q}^n \)
Unit ball \( B^n \).

In the remaining of today's lecture we \( p=2 \) Euclidean distance.

Volume of \( \mathbb{Q}^n \) : 
\[
\text{Volume of } \mathbb{Q}^n = 1^n = 1
\]

Volume of \( B^n \) : 
\[
\text{Volume of } B^n = \frac{\pi^{n/2}}{\Gamma(n/2)} = \frac{\pi^{n/2}}{\Gamma(n/2)} \prod_{i=0}^{n/2-1} (n-2i)
\]

\[
\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt
\]

\( \Gamma(n) = (n-1)! \)

Stirling formula: 
\[
\alpha! \approx \frac{\alpha^{\alpha} e^\alpha}{\sqrt{2\pi\alpha}}
\]

Proof for volume \( (B^n) \): Induction & Cavalieri.

\[
\begin{array}{c|c}
\text{n} & \text{Vol} (B^n) \\
\hline
1 & 2 \\
2 & \pi \\
3 & \frac{4\pi}{3} \\
4 & \text{Vol} (B^4) = 4.93 \\
5 & \text{Vol} (B^5) = 5.26 \\
6 & \text{Vol} (B^6) = 5.17 \ldots \text{hum} \\
\vdots & \vdots \\
13 & \text{Vol} (B^{13}) = 0.91!
\end{array}
\]

\[
\lim_{n \to \infty} \text{Vol} (B^n) = 0
\]
Volume of ball of radius $r$:

$$V_{B^m}(r) = \frac{\pi^{m/2} \cdot r^m}{\Gamma(m/2 + 1)}$$

By recurrence:

$$V_{B^m}(r) = V_{B^{m-2}}(r) \cdot \frac{\pi}{n-2} \cdot r^2$$

$V(1) = 2, V(2) = \pi$.

Question: What is the volume of the shell inside the unit ball outside the ball of radius 0.9?

In $\mathbb{R}^1$, volume:

$$= 0.2 = 0.1$$

In $\mathbb{R}^2$, volume:

$$= \frac{\pi - \pi (0.9)^2}{\pi} = \frac{\pi (1 - 0.81)}{\pi} = \frac{\pi (0.19)}{\pi} = 0.19$$

$V_{B^m}(0.9) = \frac{\pi^{m/2} \cdot (0.9)^m}{\Gamma(m/2 + 1)}$

$$\frac{V_{B^m}(0.9)}{V(B^n(1))} = (0.9)^n$$

Fraction of volume of shell:

$$1 - (0.9)^n = \frac{V_B(1) - V_B(0.9)}{V_B(1)} = 1 - \frac{V_B(0.9)}{V_B(1)}$$

If $n = 20$, fraction = 0.88 (90% volume is in a shell of radius 10%).
Size of the cube of sides 1 in \( \mathbb{R}^n \): largest diagonal has size \( \sqrt{n} \).

Size of the ball of radius \( r \) and volume 1 in \( \mathbb{R}^n \).

The radius of this ball grows like \( \sqrt{n} \).

**Proof:**

Stirling formula: \( m! \sim \frac{m^m}{e^m} \sqrt{2\pi m} \)

so \( \frac{m/2}{m} \sim \frac{(m/2)^{m/2}}{e^{m/2}} \sqrt{2\pi m} \)

and \( V(\mathbb{B}^n(r)) \sim \frac{\pi^{m/2} e^{m/2}}{(m/2)^{m/2} \sqrt{\pi m}} = \frac{e^{m/2} (\log(e\pi) + 2 \log r)}{e^{m/2} (\log m/2 + \log(2\pi))} \)

if \( V(\mathbb{B}^n(r)) \sim 1 \) we need to have

\[
\frac{m}{2} \left\{ 2 \log r - \log(m/2) + \log(e\pi) - \frac{\log(2\pi)}{m} \right\} \sim 0
\]

\[
2 \log r(n) - \log(n) + \log(2\pi e) - \frac{\log(2\pi)}{m} = \frac{1}{m} \varepsilon(m) \quad \text{as} \quad m \to \infty
\]

\[
2 \log r(n) = \log\left(\frac{m}{2\pi e}\right) + \gamma(n)
\]

\[
\log r^2(n) = \log\left(\frac{m}{2\pi e}\right) + \gamma(n)
\]

\[
r(n) \sim \frac{\sqrt{m}}{2\pi e} = 0.24 \sqrt{m}
\]
Similar phenomenon with the Gaussian.

In 1-dimensional, 90% of volume in $[-1.65, 1.65]$

In 10-dimensional, 99% of volume in the sphere of radius 1.65.

The volume is in the tail.

In fact, the volume is concentrated around the shell of radius $\sqrt{n}$.

$p$-dimensional Gaussian $\frac{f(x)}{p!} = \frac{1}{(2\pi)^{p/2}} e^{-\|x\|^2/2}$

$P\left( \frac{f(x)}{f(0)} \geq \frac{1}{100} \right) = P(X_{n-1} \leq -2 \ln \frac{1}{100}) = P(X_{n-1} \leq 2 \ln 100)$

|
|---|
|Not in 99% outer contour|

If $n = 20$ $P(X_{19} \leq 2 \ln 100) = 0.52$

$n = 10$ $P(X_{9} \leq 2 \ln 100) = 0.5$

Angles are either 0 or $\frac{n}{2}$

Hypercube $[-1, 1]^n$ diagonal = vector from the center to a corner = $(\pm 1, \pm 1, \ldots, \pm 1)$

Example: canonical basis $[0 \ldots 0, 1, 0 \ldots 0]$.

$cos \theta = \frac{\langle e, v \rangle}{\|e\| \|v\|} = \frac{\pm 1}{\sqrt{n}} \to 0$ as $n$ gets large.
Thus \( \lambda \leq \cos \pi \).

Any set of points lying near a diagonal is projected at \( 0 \).

Any set of points along a coordinate axis will be preserved.

**Balls enclosing a ball**

\[ \mathbb{D}_n : \text{hypercube} : \text{place a ball of radius } \frac{1}{2} \text{ on each vertex}. \]

Consider the ball inside the cube just touching the \( 2^n \) balls on the vertices. As soon as \( n \geq 5 \) this ball is outside the unit cube.

Indeed, radius of the enclosed ball = radius of diagonal \( -\frac{1}{2} = \frac{1}{2} \sqrt{n} - \frac{1}{2} = \frac{1}{2} (\sqrt{n} - 1) \)

half diagonal
Volume of the $n$-dimensional sphere.

Let $V_n(r)$ be the volume of the $n$-dimensional sphere. By a homogeneity argument,

$$\frac{V_n(r)}{V_n(1)} = r^n$$

We only compute $V_n(1)$.

Consider the last coordinate $x_n$ and slice the ball along this direction.

$$V_n(1) = \int_{-1}^{1} V_{n-1}(\sqrt{1-x_n^2}) \, dx_n$$

$$= \int_{-1}^{1} V_{n-1}(1) (\sqrt{1-x_n^2})^{n-1} \, dx_n$$

$$= V_{n-1}(1) \int_{-1}^{1} (\sqrt{1-t^2})^{n-1} \, dt$$

Let $t = \cos \theta$

$$V_n(1) = V_{n-1}(1) \int_0^{\pi} (\sin \theta)^{n-1} \sin \theta \, d\theta = \int_0^{\pi} (\sin \theta)^n \, d\theta$$

Let us denote $I_n = \int_0^{\pi} \sin^n \theta \, d\theta$.

We have $V_n(1) = V_{n-1}(1) I_n$.
We have therefore

\[ V_n = V_0 \prod_{i=1}^{n} I_i \quad n \geq 1 \]

Now, if we integrate by part \( I_n \) we obtain

\[
\int_0^{\pi} \sin^n \theta \, d\theta = \int_0^{\pi} \sin^{n-1} \theta \cdot \sin \theta \, d\theta = \left[ -\cos \theta \sin^{n-1} \theta \right]_0^\pi + \int_0^{\pi} (n-1) \cos \theta \sin^{n-2} \theta \cos \theta \, d\theta
\]

\[
= (n-1) \int_0^{\pi} \cos^2 \theta \sin^{n-2} \theta \, d\theta = (n-1) \int_0^{\pi} (1 - \sin^2 \theta) \sin^{n-2} \theta \, d\theta
\]

\[
= (n-1) \left( \int_0^{\pi} \sin^{n-2} \theta \, d\theta - \int_0^{\pi} \sin^n \theta \, d\theta \right)
\]

\[
= (n-1) \int_0^{\pi} \sin^{n-2} \theta \, d\theta - (n-1) \int_0^{\pi} \sin^{n} \theta \, d\theta
\]

\[
= I_{n-2} - (n-1) I_n
\]

So

\[
I_n = - (n-1) I_n + (n-1) I_{n-2}
\]

or

\[
n I_n = (n-1) I_{n-2}
\]

or

\[
I_n = \frac{n-1}{n} I_{n-2}
\]

if \( n = 2p \)

\[
I_{2p} = \frac{2p-1}{2p} \cdot \frac{2p-3}{2p-2} \cdot \ldots \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} I_0
\]

\[
I_{2p-1} = \frac{2p-2}{2p-1} \cdot \ldots \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot I_1
\]

\[
I_0 = \pi \quad \text{and} \quad I_1 = 2
\]

Also

\[
I_{2p-1} I_{2p} = \frac{1}{2p} \cdot I_0 I_1 = \frac{2\pi}{2p}
\]
for \( n = 2p \)

\[
V_{2p} = V_0 \frac{\pi}{2p} \frac{\pi}{2p-2} \cdots \frac{\pi}{2} = V_0 \frac{\pi^p 2^p}{2^p \cdot p!} = \frac{\pi^p}{p!} = \frac{\pi^p}{\Gamma(p+1)}
\]

Similarly, we can show

\[
I_{2p} \cdot I_{2p} = \frac{2\pi}{2p+1}
\]

and

\[
V_{2p} = V_0 \frac{\pi}{2p} \frac{\pi}{2p-2} \cdots \frac{\pi}{2} = \frac{2^p \pi^p}{(2p+1)(2p-1) \cdots 3 \cdot 1}
\]

\[
= \frac{\pi^p}{(p+1/2)(p-1/2) \cdots 3/2 \cdot 1/2} = \frac{\pi^p}{\Gamma(p+3/2)}
\]

**In summary**

\[
V(n) = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)}
\]

By Stirling formula

\[
\Gamma(n/2 + 1) \sim \sqrt{2\pi n} e^{-n/2} (n/2)^{(n+1)/2}
\]

And

\[
V_n(1) \sim \left( \frac{2\pi e}{n} \right)^n
\]
Relationship between volume and surface area of the ball of radius $R$.

$V_n(R) =$ volume of ball of radius $R$

$$ = \int_0^R \sigma_n(r) \, dr$$

Surface of the sphere of radius $R$ in dimension $n$

$\sigma_n(r) = r^{n-1} \sigma_m(1)$ because the surface of the sphere is a homogeneous function of degree $n-1$

so $V_n(R) = \int_0^R \sigma_m(1) r^{n-1} \, dr = \sigma_m(1) \frac{R^n}{n}$

For $R=1$ we get

$\sigma_n(1) = \frac{n \pi^{n/2}}{\Gamma(n/2+1)}$. 