Definition
The set of all subspaces of dimension $m < n$ is called the Grassmannian, $G(m, n)$.

Lemma
$G(m, n)$ is homeomorphic to $O(n)/O(m) \times O(n-m)$.

Idea of the proof:
Let $L \in G(m, n)$, $u_1, \ldots, u_m$ be orthonormal basis of $L$.

We can construct $U \in O(n)$ such that

$$Ue_i = u_i, \quad i = 1, \ldots, m.$$  

We want to collect all $U$ that map $e_i$ to $u_i$ into the same clan: $U$ and $V$ both map $e_1, \ldots, e_m$ to $u_1, \ldots, u_m$ if there exists a matrix $W \in O(n-m)$ and a matrix $T \in O(m)$ such that

$$U = V \begin{bmatrix} I_m & 0 \\ 0 & \begin{bmatrix} I_{n-m} & W \\ 0 & \begin{bmatrix} m & n \\ 0 & \end{bmatrix} \end{bmatrix} h_{n-m} = V \begin{bmatrix} T & 0 \\ 0 & W \end{bmatrix}.$$
A rotation on the first $m$ coordinates before landing on the last $n - m$.

**Consequence of the Lemma:**
The Haar measure on $G(m, n)$ is invariant with respect to the action of $O(n)$.

In the following we construct the Haar measure $\Theta$ on $G(m, n)$.

Let us start with $m = 1$.
If $n = 2$, we can measure a set $B \subseteq G(m, n)$ by measuring the intercept with the unit circle.

If $n \geq 2$, we can measure the trace of $B$ on the unit sphere. We should only take one hemisphere.

In general, we observe that $O(n)$ acts transitively on $G(m, n)$: for any two elements $E, F \in G(m, n)$, there exists $U \in O(n)$ such that $F = UE$.

Indeed, we can consider a basis of orthonormal vectors $e_1, \ldots, e_m$ of $E$ and $v_1, \ldots, v_m$ of $F$ and define $U$ so that $Ue_i = v_i$. 
Because $O(n)$ acts transitively on $G(m,n)$ we can define the Haar measure from the Haar measure on $O(n)$.

**Theorem**

Let $A \in G(m,n)$. The pushforward of the Haar measure on $O(n)$ by the map

$$U \in O(n) \quad \mapsto \quad U L_0 \in G(m,n)$$

for a given $L_0$ (any $L_0$) in $G(m,n)$,

is the Haar measure on $G(m,n)$.

$$\Theta(A) = \mathcal{V} \{ U \in O(n), \quad U L_0 \in A \}$$

Haar measure on $O(n)$.

**Proof**

$\Theta$ is invariant under the action of $O(n)$.

$\forall A \in G(m,n), \forall U \in O(n)$

$$\Theta(UA) = \Theta(A).$$

Indeed

$$\Theta(UA) = \mathcal{V} \{ V \in O(n), \quad VL_0 \in UA \} = \mathcal{V} \{ V \in O(n), \quad UVL_0 \in A \} = \mathcal{V} \{ W \in O(n), \quad WL_0 \in A \}$$
Remark: we can define a metric on \( G(m,n) \).
Consider the projection on \( E \in G(m,n) : P_E \)
Distance between \( E \) and \( F = d(E,F) = \| P_E - P_F \| \)
operator norm.

the action of \( O(n) \) preserves the distance between \( E \) and \( F \).

Interpretation of the theorem about the Haar measure.

In order to sample \( G(m,n) \): pick a subspace \( L \in G(m,n) \).
= origin subspace
rotate \( L \) until it covers \( A \in G(m,n) \): measure of \( A \)
= amount of rotation.

Remark:
there is a characterization of \( G(m,n) \) as a quotient of the subsets m-tuples that are linearly independent.

\( \mathcal{U} = \) subspace of m-tuples \((u_1, \ldots, u_m)\) that are linearly independent.
Define equivalence relation: \((u_1, \ldots, u_m) \sim (v_1, \ldots, v_m)\) if
\( \exists A \in GL(m) \) that maps \( u_i \) to \( v_i \). Then
\( G(m,n) = \mathcal{U} / \sim \) quotient space.