ECEN 3400, Fall 2005 Midterm 2  
Tuesday Oct 25, 6-8 PM

NAME: Solution

Structure of the exam

• No notes, books or calculators allowed
• Except for problem 1, work neatly on separate sheets of paper and show your work.
• There are 10 total problems – clearly the intended solutions to each are short.
• If you are writing down 20-line solutions, stop and think – you’ve made a wrong turn.
• If a problem says you do not need to do some operation, don’t.

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ECEN 3400 Equation sheet (Ch12-17)

**Magnetic force**
\[ d\mathbf{F}_{12} = I_2 d\mathbf{l}_2 \times \left( \frac{\mu_0}{4\pi} \frac{l_1 d\mathbf{l}_1 \times \mathbf{u}_r}{r^2} \right) \quad [N] \]
\[ \mu_0 = 4\pi \times 10^{-7} \quad [H/m] \]
\[ = I_2 d\mathbf{l}_2 \times \mathbf{B} \]
\[ \mathbf{F} = Q \mathbf{v} \times \mathbf{B} \]

**Biot-Savart Law**
\[ \mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{l_1 d\mathbf{l}_1 \times \mathbf{u}_r}{r^2} \quad [T] \]

**Magnetic flux**
\[ \Phi = \int_S \mathbf{B} \cdot d\mathbf{S} \quad [Wb] \]

**Conservation of Magnetic flux**
\[ \Phi_{\text{closed}} = \oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \]

**Ampere’s Law (time-invariant)**
\[ \oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \oint_S \mathbf{J}_\text{total} \cdot d\mathbf{S} = \mu_0 I_{\text{total–enc}} \]
\[ \oint_C \mathbf{H} \cdot d\mathbf{l} = \oint_S \mathbf{J}_\text{applied} \cdot d\mathbf{S} = I_{\text{applied–enc}} \]
\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}_\text{total} \quad \nabla \times \mathbf{H} = \mathbf{J}_\text{applied} \quad \nabla \times \mathbf{M} = \mathbf{J}_\text{magnetization} \]

**Definition of H, M**
\[ \mathbf{B} = \mu_0 \left( \mathbf{H} \left[ \frac{A}{m} \right] + \mathbf{M} \left[ \frac{A}{m} \right] \right) = \mu_0 \left( 1 + \chi_m \right) \mathbf{H} = \mu_0 \mu_r \mathbf{H} = \mu \mathbf{H} \]

**Boundary conditions**
\[ H_{1-\text{tangential}} - H_{2-\text{tangential}} = 0 \]
\[ B_{1-\text{normal}} - B_{2-\text{normal}} = 0 \]
\[ J_{\text{magnetization–surface}} = \mathbf{n} \times (\mathbf{M}_1 - \mathbf{M}_2) \quad [\frac{A}{m}] \]

**Faraday’s law**
\[ e = \oint_C \mathbf{E}_{\text{ind}} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} \quad [V] \]

**Inductance**
\[ \Phi_{\text{due to } l_{12}} = L_{12} \left( H \right)_1 \left[ A \right] \]
\[ e_{12} = -\frac{d\Phi_{12}}{dt} = -L_{12} \frac{di_1}{dt} \]
\[ \Phi_{\text{self}} \left[ \mathbf{Wb} \right] = L \left( H \right)_1 \left[ A \right] \]
\[ e = -\frac{d\Phi_{\text{self}}}{dt} = -L \frac{di}{dt} \]

**Magnetic Energy**
\[ W_m \left[ J \right] = \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n L_{jk} I_j I_k = \frac{1}{2} L I^2 \quad \text{(if } n = 1) \]

**Energy density**
\[ W_m \left[ \frac{1}{2} \frac{\mathbf{B}^2}{\mu} \right] = \frac{1}{2} \mu H^2 = \frac{1}{2} BH \]

**Force on body**
\[ F_x = \frac{dW_m(x)}{dx} \quad \left| _{\Phi=\text{constant}} \right. = \left. +\frac{dW_m(x)}{dx} \right| _{l=\text{constant}} \]
1a) The figure to the left shows a current loop in the XY plane with current circulating in a right-hand sense around the Y axis. The diagram on the right shows the YZ cross section through this magnetic dipole. Current is coming out of the page on top and into the page on the bottom. Sketch ~10 lines of magnetic flux density inside the square (an imaginary boundary). No other currents or materials are nearby. Remember to indicate direction with occasional arrow heads. Try to show how the field varies throughout the region.
The figure below shows the same current loop that was used in part (a). The current \( I_1 \) is increasing with time.

We now wish to use this current loop to create an induced current in the loops A, B, and C, positioned as shown. A is closest to the current at the origin, then B, then C. The arrows on loops A, B, and C give the sign convention for current in those loops (that is, a current traveling with the arrows is defined as positive). Given only the information provided, write down as much information as possible about:

1b) The sign or value of the current in each loop A, B, C.

*There is almost no flux through A, so \( I_A > 0 \) or \( I_A \sim 0 \) are both OK answers.*

\( I_B \) and \( I_C \) are defined in the same direction as \( I \) which is increasing, so the sign of the current in both must be negative (by Lentz’ Law, or if you prefer, to keep the universe from blowing up due to positive feedback).

1c) The relative magnitude of the current in each loop. That is, how do \(|I_A|\), \(|I_B|\), and \(|I_C|\) compare?

*I intended \( I_A \sim 0 \), but actually didn’t set the problem up quite right, so ignored \( I_A \) in this part. Loop C is farther from the source so must have a smaller flux. Thus \(|I_C| < |I_B|\).*
ECEN 3400 Midterm 2, 10/25/05
Problem 2

Hall-effect magnetic field meter.

A current flow formed from positive charge carriers of charge Q (e.g. “holes” in a semiconductor) is forced through the volume \((a \times a \times L)\) shown below. Through some other measurement, the velocity of these charges is known to be \(v\) [m/s]. A voltage \(V_p\) is measured across the top and bottom metal plates, as shown, indicating the presence of a magnetic field.

2a) What is the orientation of the magnetic flux density, \(B\)?

Positive particles are being deflected upward, thus by the right-hand rule, \(B\) must be in \(-y\) direction.

2b) What is the magnitude of the \(B\) field in terms of the quantities given?

The particles are experiencing a force due to the magnetic field.

\[
F_z = QvB \quad [N]
\]

This force will cause to accumulate on the metal plates, setting up an electric field. The charges will continue to accumulate until the force due to the \(E\) field balances the force due to the magnetic field. Thus the \(E\) field in \(z\) is equal to the magnetic force divided by the charge and the voltage is simply this \(E\) field times the thickness of the plate, \(a\).

\[
E_z = \frac{F_z}{Q} = \frac{vB}{C} \quad \left[\frac{N}{C}\right]
\]

\[
V_p = \int_0^a E_z \, dz = vBa \quad [V]
\]
3a) Given a current $I_{\text{wire}}(t)$ traveling in the $+z$ direction through a very long, very thin wire, calculate the magnetic flux density in the space near the wire. Be sure to include both magnitude and direction in your answer.

$$\oint_C \mathbf{B} \cdot d\mathbf{a} = \mu_0 I_{\text{total-enc}}$$  \hspace{1cm} \text{Ampere's law} $$B_\phi(2\pi r) = \mu_0 I_{\text{wire}}$$ $$B_\phi = \frac{\mu_0 I_{\text{wire}}}{2\pi r} \quad [T]$$

3b) A perfectly conducting square wire loop of side length $a$ is placed a distance $b$ from the wire. Using your result from part (a), write an expression for the flux through the loop. You do \textit{not} need to carry out the integral, but write a complete expression including sign as defined by the arrow on the loop perimeter (labeled $I_{\text{loop}}$).

$$\Phi = \int \mathbf{B} \cdot d\mathbf{S}$$
$$= \int_0^b dr \int_a^{a+b} \frac{\mu_0 I_{\text{wire}}}{2\pi r} \quad [Wb]$$

Many people attempted to do the integral. Sigh.
4a) A single resistor $R \, [\Omega]$ is placed in the loop as shown. What is the current $I_{\text{loop}}$? You may use results from the problem 3 as given (that is, you may write your answer to problem 4 in terms of a magnetic flux density or a magnetic flux but do not need explicit forms for these quantities).

$$I_{\text{loop}} = \frac{V}{R} = -\frac{1}{R} \frac{d\Phi}{dt} \quad [A]$$

4b) Given the equipment you use in the lab, describe how you would measure the mutual inductance between the wire and the loop. List the required steps and any calculations needed.

$$e_{12} = -L_{12} \frac{di_1}{dt} \quad [V]$$

*Hook a current source to the wire with peak amplitude $I$ and sinusoidal radian frequency $\omega$. Measure $V$ across the resistor using a scope with + lead on bottom and -lead on top. $L_{12}$ is then $V / (\omega I)$.*
Two air-core solenoids of radius $a$ with identical winding directions as shown are separated by a small gap. The flux density in the region of this gap is well approximated by that of an infinite solenoid. A sheet of initially unmagnetized ($\mathbf{M} = 0$) iron of radius $b>a$ is introduced into the gap and the current $I$ is turned on. Assume that the resulting flux density is sufficiently large that the iron is driven to saturation. That is, there is only one domain wherever the iron is within the $\mathbf{B}$ field.

5a) Describe the distribution of magnetic dipoles $\mathbf{m}$ within the iron. If there are different areas with different properties, describe each including its location and the orientation of magnetic dipoles within.

Outside the radius $a$, the magnetic dipoles are randomly distributed. Inside $a$, the magnetic dipoles are in the $-\mathbf{x}$ direction.

5b) Using the result of part (a), what is the distribution of the magnetization vector $\mathbf{M}$ within the iron? You do not need to calculate the value for $\mathbf{M}$, only indicate its direction.

$\mathbf{M}$ is zero for $r>a$ and $\mathbf{M}$ is in the $-\mathbf{x}$ direction for $r<a$.

5c) Is the on-axis magnitude of the $\mathbf{B}$ field in the iron greater, the same, or less than the on-axis magnitude of $\mathbf{B}$ inside the solenoids?

$\mathbf{B}$ is greater in the iron due to the magnetization of the iron. That is, all the magnetic dipoles $\mathbf{m}$ and the magnetization vector $\mathbf{M}$ are aligned parallel to $\mathbf{B}$ and thus add. ½ credit for $=$ since this is what B.C. tells you. BC doesn’t know about edges of iron.

5d) Using the previous results, describe the distribution of magnetization current. You can either derive this, or instead use logic and the distribution of magnetic dipoles from part (a) to describe it. The important features to mention are the spatial locations and directions of any magnetization currents.

There is a magnetic surface current density $\mathbf{J}_s$ at $r=a$. It is in the same direction as the solenoid currents (it must be to add to $\mathbf{B}$) which is in the $-\phi$ direction.

OR

Small current loops same direction as I set up in iron. All cancel except at $r=a$ so must have surface current density that looks like solenoid at $r=a$.

OR

$\mathbf{B}$ in iron bigger $r<a$, zero $r>a$, so must have solenoid-like current at $r=a$ to make this $\mathbf{B}$. 
A betatron is pictured below. An evacuated tube of radius $b$ contains free electrons. Current in the solenoids is increased linearly with time according to:

$$I(t) = \gamma t$$

The solenoids have $N'$ turns per unit length.

What is the acceleration of the electrons of charge $Q_e$ and mass $m_e$? Don’t worry about the sign of the acceleration (too many negatives to keep track of here), just the magnitude will do.

$$e = 2\pi b E_{ind} = -\frac{d\Phi}{dt} = -\frac{dB}{dt}\pi a^2 = -\mu_0 N' \frac{dl}{dt}\pi a^2 = -\mu_0 N' \gamma [V]$$  \text{ Faraday’s law.}

$$E_{ind} = -\frac{\mu_0 N' \gamma}{2\pi b} \pi a^2 \left[\frac{N}{C}\right]$$

$$F = mA \left[N\right]$$  \text{ Calculate $E$ from emf}

$$A = \frac{F}{m} = \frac{E_{ind}(-Q_e)}{m_e} = \frac{\mu_0 N' \gamma a^2 Q_e}{2b m_e} \left[\frac{m}{s^2}\right]$$  \text{ Newton’s law}

Solve for $A$. 

\[ 
\begin{align*}
196 x & = \pi a^2 (V) \\
200 y & = \pi a^2 (V) \\
214 z & = \pi a^2 (V)
\end{align*}
\]
A rectangular area of surface current density $J_s$ [A/m] is flowing in the XY plane in the $+u_x$ direction. Don’t worry about the fact that this current must be coming from and going to somewhere, just consider this rectangular area of dimensions $a$ by $b$ as shown.

7a) Write down the expression for an infinitesimal unit of magnetic flux density at the point $P$ caused by an infinitesimal unit of current. This should be a complete mathematical expression with explicit expressions for any vectors or other quantities needed. If your expression contains a cross product, carry it out. Hint: This last step should take exactly one line.

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I_s dy \times u_x}{r^2} \quad [T]$$  \hspace{1cm} \text{Biot-Savart}$$

$$= \frac{\mu_0}{4\pi} \frac{(J_s dy)(dxu_z) \times (-xu_x - yu_y + P_z u_z)}{(x^2 + y^2 + P_z^2)^{\frac{3}{2}}} \quad \text{Put in appropriate bits}$$

$$= \frac{\mu_0}{4\pi} \frac{J_s dy dx}{(x^2 + y^2 + P_z^2)^{\frac{3}{2}}} (-yu_z - P_z u_y) \quad \text{Take cross product}$$

7b) Given the result of part a, write down an integral expression for the magnetic flux density at $P$. You do not have to recopy the results of (a) and in fact can do part (b) without the results of part (a). That is, your answer to (b) can simply use a symbol for the results of (a).

$$\mathbf{B} = \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} d\mathbf{B} \quad [T]$$
ECEN 3400 Midterm 2, 10/25/05
Problems 8 & 9

In a student lab experiment, a current source (marked with an “i”) drives a current \( i(t) \) into a solenoid of length \( L_{\text{sol}} \) [m], \( N \) turns and radius \( a \). The core of the solenoid is a wooden dowel. The current is

\[ i(t) = I \sin(\omega t) \]

8) Using the definition of self-inductance \( e = -L \frac{di}{dt} \), what is the inductance \( L \) [H] of the solenoid?

\[
L = \frac{e}{-di/dt} = -\frac{N(d\Phi/dt)}{-di/dt} = N\mu_0 \frac{N}{L_{\text{sol}}} \frac{di}{dt} \pi a_2 = \mu_0 \frac{N^2}{L_{\text{sol}}} \pi a_2 \quad [H]
\]

Note one \( N \) from \( B \), one from integrating around coil. Many people missed second one.

If the inductance \( L \) is given (you don’t need the results of 8) consider the two lab benches below. Both benches have identical solenoids driven by identical current sources as described above. Each student uses a oscilloscope to measure the voltage across the coil (at the point marked with \( V_L \) or \( V_R \)). The student on the right-hand bench has a scope with one longer lead, so lays it on the table in a loop of radius \( b \), as shown.

9a) What voltage \( V_L(t) \) does the left-hand student measure?

\[ V_L(t) = L \frac{di}{dt} = \mu_0 \frac{N^2}{L_{\text{sol}}} \omega \cos(\omega t) \quad [V] \]

9b) Does the right-hand student measure a different voltage? If so, by how much. Don’t worry about the sign of the difference.

The probe makes a loop around the coil and thus picks up an emf. Via Faraday

\[ e_{\text{lead}} = -\frac{d\Phi}{dt} = -\mu_0 \frac{N}{L_{\text{sol}}} \omega \cos(\omega t) \quad [V] \]

OR, note that just one extra turn on emf, so \( V_R = (N+1)/N \ V_L \)
The figure below shows an iron rod of relative permeability $\mu_r$ partially inserted in a solenoidal coil. The coil has $N'$ turns per unit length and is $L_{sol}$ long. A constant current $I$ is being driven into the coil.

Assume the magnetic field in the solenoid may be taken as uniform inside its cylindrical volume and zero everywhere outside including directly beyond the coil ends.

Find the force on the iron rod in the x direction. Hint: You can use the results of previous problems to solve this one (if you trust them) but it is not necessary and is about the same amount of effort either way.

**Key observations:** 1) There are two volumes within the solenoid with two different energies. The volume of each will depend on $x$, from which we’ll see a force on the body.

\[
\begin{align*}
\text{From energy density:} \\
W_m \left[ \frac{L_{sol}}{a} \right] &= \frac{1}{2} \mu_0 H^2 = \frac{1}{2} \mu_0 (N' I)^2 \\
W_m [J] &= \pi a^2 \left[ \frac{1}{2} \mu_0 (N' I)^2 x + \frac{1}{2} \mu (N' I)^2 (L_{sol} - x) \right] \\
F_x &= \frac{dW_m}{dx} = \pi a^2 \frac{1}{2} \mu_0 (N' I)^2 (1 - \mu_r) [N]
\end{align*}
\]

Key thing to note here is that $w_m$ is energy density, so $x$ comes into equation through multiplication by the volume of the two inductors.

\[
\begin{align*}
\text{From energy in an inductor:} \\
L &= \mu_0 N' N \pi a^2 \\
&= N' \pi a^2 \left[ \mu_0 N' x + \mu (N' I)^2 (L_{sol} - x) \right] \\
W_m &= \frac{1}{2} LI^2 \\
&= \pi a^2 \left[ \frac{1}{2} \mu_0 (N' I)^2 x + \frac{1}{2} \mu (N' I)^2 (L_{sol} - x) \right]
\end{align*}
\]

And take the derivative as on the left. Key observation here is again to treat as two inductors of differing lengths.

**Note that force is negative since $\mu_r > 1$. If $\mu_r = 1$, force equals 0, so you can’t pick up nonmagnetic materials with an electromagnet.**