9

Energy, Forces, and Pressure in the Electrostatic Field

9.1 Introduction

Measured by average human standards, electric energy, forces, and pressures are small. For example, it is virtually impossible to have an electric force of magnitude greater than a few newtons, or electric systems with energy exceeding a few thousand joules. Nevertheless, electric forces have surprisingly wide engineering applications. For example, purification of some ores, extraction of solid particles from smoke or dusty air, spreading of the toner in xerographic copying machines, and efficient and economical painting of car bodies are all based on electric forces.

Electrostatic energy is of equal engineering importance. For example, sufficient energy to destroy virtually any semiconductor device can easily be created in the field of a person charged by walking on a carpet. This is the meaning of the commonly used warning “static sensitive.”

Questions and problems: Q9.1
9.2 Energy of a Charged Capacitor

In the preceding chapter, we defined capacitance and described and analyzed several types of capacitors. It is easy to understand that every charged capacitor contains a certain amount of energy. For example, the plates of a charged parallel-plate capacitor attract each other. If we let them move, they will perform a certain amount of work. In order for a system to do work, it must contain energy. Since the capacitor plates do not attract each other if they are not charged, it follows that some energy is stored in a charged capacitor. We can find how much energy there is by looking at what happens while a capacitor is being charged.

Consider a capacitor of capacitance $C$ that is initially not charged. We wish to charge its electrodes with $Q$ and $-Q$. To do this, we take small positive charges $dq$ from the negative electrode and take them over to the positive electrode. To move the charge against the electric forces ($dq$ is attracted by the negative electrode, and repelled by the positive electrode), we must do some work. Suppose that, at an instant during this process, the capacitor electrodes are charged with charges $q$ and $-q$ ($0 < q \leq Q$). This means that the potential difference between them is $v = q/C$. By definition of the potential difference between the electrodes, the work we have to do against the electric forces in moving the next $dq$ from the negative to the positive electrode equals $dA = v dq = q dq/C$. So the total work that needs to be done to charge the capacitor electrodes with the desired charges, $Q$ and $-Q$, is

$$A = \int_{0}^{Q} \frac{q}{C} dq = \frac{Q^2}{2C} \quad (J). \quad (9.1)$$

Since there were no losses in charging the capacitor, this work was transformed into potential energy of the capacitor. This energy we call the electric energy. Noting that $Q = CV$, the electric energy of a charged capacitor is thus given by the following equivalent expressions:

$$W_e = \frac{Q^2}{2C} = \frac{1}{2} QV = \frac{1}{2} CV^2 \quad (J). \quad (9.2)$$

(Energy of a charged capacitor)

Let us look at a few examples. The largest possible energy of an air-filled parallel-plate capacitor with plate area $S = 1 \, \text{dm}^2$ and with a distance between plates of $d = 1 \, \text{cm}$ is

$$(W_e)_{\text{max}} = \frac{1}{2} \epsilon_0 \frac{S^2}{d} E_{\text{max}}^2 d^2 = \frac{1}{2} \epsilon_0 E_{\text{max}}^2 S d \simeq 4 \, \text{mJ}, \quad (9.3)$$

since $E_{\text{max}} \simeq 30 \, \text{kV/cm}$. (The maximum energy corresponds to the maximum voltage, i.e., to the maximum electric field.) This is not very much energy from a human viewpoint (although it can destroy practically any semiconductor device).
If we consider a high-voltage capacitor, for example one where \( V = 10 \text{kV} \) and \( C = 1 \mu\text{F} \), we obtain instead
\[
W_e = \frac{1}{2} CV^2 = 50\text{J}.
\] (9.4)

This is roughly equivalent to the potential energy of a 1-kg coconut that is 5 m above ground. The energy of high-voltage capacitors is clearly quite large, and touching their electrodes can be fatal.

Questions and problems: Q9.2 to Q9.8, P9.1 to P9.4

9.3 Energy Density in the Electrostatic Field

The expression for the energy of a parallel-plate capacitor can be rewritten as
\[
W_e = \frac{1}{2} CV^2 = \frac{1}{2} \varepsilon \frac{S}{d} V^2 = \frac{1}{2} \varepsilon E^2 S d,
\] (9.5)

since \( V/d = E \). The product \( Sd \) is equal to the volume of the capacitor dielectric (i.e., the volume of the domain with the field). Therefore, no error will be made in computing the capacitor energy if we assume that it is distributed in the entire field, with a density

\[
w_e = \frac{W_e}{v} = \frac{1}{2} \varepsilon E^2.
\] (9.6)

(Energy density, J/m³, in an electrostatic field)

We will now show that this result is valid in general and not just for a parallel-plate capacitor. Let us look at a system of charged bodies in an arbitrary dielectric, as shown in Fig. 9.1. When we place thin aluminum foil exactly over an equipotential surface, we do not change the electric field. This is because we place a conducting surface, which must be equipotential, on an equipotential surface.

We can therefore place many thin aluminum foils on many equipotential surfaces, very close to each other, without changing the field. However, in this way we have divided up the space around the charged bodies into a very large number of small parallel-plate capacitors. The total energy of this system is given by the sum of all the little capacitor energies. The energy density of each of the capacitors is equal to \( w_e = \varepsilon E^2 / 2 \), where \( E \) is the electric field at that point, and \( \varepsilon \) the permittivity at that point. Consequently the energy of the whole system is given by
\[
W_e = \int \frac{1}{2} \varepsilon E^2 \, dv.
\] (9.7)

(Energy of an electric field)
Figure 9.1 The electric field does not change when two aluminum foils are placed exactly at two close equipotential surfaces. Charges are induced on the surface of the foils (as shown in the enlarged circle), and the field between the foils is approximately uniform.

The integral in the equation is a volume integral over the entire volume in which the electric field exists.

**Example 9.1—Energy of a high-voltage coaxial cable.** In Example 8.8, we saw that a high-voltage coaxial cable consists of two dielectric layers and that the electric field in the two layers is given by

\[ E_1 = \frac{Q'}{2\pi \varepsilon_1 r} \quad a < r < b \]
\[ E_2 = \frac{Q'}{2\pi \varepsilon_2 r} \quad b < r < c. \]

The energy per unit length of the cable is the sum of the energies contained in the two dielectric layers:

\[ W_e' = \int_{\text{layer 1}} \frac{1}{2} \varepsilon_1 E_1^2 dv' + \int_{\text{layer 2}} \frac{1}{2} \varepsilon_1 E_2^2 dv'. \]

Now \( dv' = 2\pi r dr \), and \( E_1 \) and \( E_2 \) are given by the expressions at the beginning of the example. With respect to \( r \), the first integral has limits from \( a \) to \( b \), and the second one from \( b \) to \( c \). After integrating, we get

\[ W_e' = \frac{Q'^2}{2} \left[ \frac{\ln(b/a)}{2\pi \varepsilon_1} + \frac{\ln(c/b)}{2\pi \varepsilon_2} \right]. \]

If we use \( W_e = \frac{\varepsilon_0 E^2}{2} \), we get the same expression.

\[ = \left( \frac{Q'}{2e} \right)^2 / a \]
We have concluded that energy contained in an electrostatic system can be determined if we assume that it is distributed throughout the field, with a density given in Eq. (9.6), even if the dielectric is a vacuum. In the case of dielectrics in the field, obviously at least some of the energy must be stored throughout the dielectric: to polarize the dielectric, the electric field needs to do some work at the very point where a dielectric molecule is, and this molecule acquires some energy. This means that the energy used to polarize a dielectric is distributed throughout the dielectric, just like the energy used in stretching a spring is distributed inside the entire spring.

In the case of a vacuum, however, such a physical explanation does not exist. How can we then state that the field in a vacuum also contains energy? In electrostatics, such a proof is not possible, but we shall see that in time-varying fields, the field does have energy distributed in a vacuum. For example, we know that a radio wave, which is but a combination of electric and magnetic fields, is able to carry a signal from the earth to Jupiter and back. This is a vast distance, and for a significant time the signal is neither on earth nor on Jupiter. It travels through a vacuum in between. It certainly carries some energy during this travel, because we are able to detect it.

Questions and problems: Q9.9 to Q9.11, P9.5 to P9.12

9.4 Forces in Electrostatics

We started discussing electrostatics with Coulomb’s law for the electric force between two point charges. Because the principle of superposition applies, it can be used as a basis for determining the electric force on any body in a system where we know the distribution of charges.

As an example, consider the two charged conducting bodies shown in Fig. 9.2, with a known surface charge distribution. Let us find the expression for the force $F_{12}$ with which body 1 acts on body 2. To find this force, we divide body 2 into small patches $dS_2$ and determine the electric field strength $E_1$ at all these patches due to the charge on body 1. The force is then obtained as

$$F_{12} = \int_{S_2} \sigma_2 \, dS_2 E_1.$$  \hspace{1cm} (9.8)

![Figure 9.2 Finding the electric force between two large charged conducting bodies](image-url)
In this equation, the field $E_1$ is given by

$$E_1 = \frac{1}{4\pi \varepsilon_0} \int_{S_1} \frac{\sigma_1 \, dS_1}{r_{12}^2} \, \mathbf{u}_{r_{12}},$$  \hspace{1cm} (9.9)$$

where $\mathbf{r}_{12}$ is the vector directed from an element of body 1 toward an element of body 2, and $\mathbf{u}_{r_{12}}$ is the unit vector along this direction.

**Example 9.2—Force between the plates of a parallel-plate capacitor.** Let us find the electric force that the electrodes of a parallel-plate capacitor of plate area $S$ exert on each other. We know that the charge is distributed practically uniformly on the electrode surface, i.e., the charge distribution is known. Let the capacitor be connected to a source of voltage $V$. The charge on the positive plate is then $Q = CV = \varepsilon_0 SV/d$. We have found by Gauss' law that the electric field strength of the charge on the positive plate at the negative plate is $E_Q = Q/(2\varepsilon_0 S)$. So using Eq. (9.8) we have

$$F_{12} = \int_S (-\sigma) \, dS E_Q = -QE_Q.$$

The force is attractive, as it should be, and its intensity is given by

$$F_{12} = QE_Q = \frac{Q^2}{2\varepsilon_0 S} = \frac{1}{2} \varepsilon_0 S \frac{V^2}{d^2}.$$

**Example 9.3—Magnitude of electric force in some typical devices.** What is the maximal electric force in a parallel-plate capacitor filled with air, with $S = 1 \text{ dm}^2$? The air breakdown field is $V/d \simeq 30 \text{ kV/cm}$, so we obtain $F_{12} \simeq 0.4 \text{ N}$, which amounts to the weight of about one quarter of a glass of water. Note that this is the largest possible force.

Another example is the force between the two wires of a two-wire line connected to a source of voltage $V$. The charge per unit length on the wires is $Q' = C'V = \pi \varepsilon_0 V / \ln(d/a)$. At the place of the negatively charged wire, the positively charged wire produces a field $E_{Q'}$ equal to

$$E_{Q'} = \frac{Q'}{2\pi \varepsilon_0 d},$$

The force per unit length on the negatively charged wire is then

$$F'_{12} = -Q'E_{Q'} = -\frac{Q'^2}{2\pi \varepsilon_0 d} = -\frac{\pi \varepsilon_0 V^2}{2d[\ln(d/a)^2]}.$$

The minus sign tells us that the force is attractive, which it should be. Its maximal value for $a = 2.5 \text{ mm}$, $d = 1 \text{ m}$, and $E_{\text{max}} = 30 \text{ kV/cm}$ is $F'_{12} \simeq 0.00313 \text{ N/m}$. This is again quite a small force.

The two preceding examples illustrate the statement in the chapter introduction that in normal circumstances, electric forces acting between charged bodies are very
small. Therefore, they can be neglected most of the time. There are nevertheless many applications of the electrostatic forces, as will be discussed in Chapter 11.

Questions and problems:  Q9.12 to Q9.19, P9.13 to P9.16

9.5 Determination of Electrostatic Forces from Energy

We saw that we can find electric forces between charged bodies only if we know the charge distribution on them, which is rarely the case. Moreover, the previously discussed method cannot be used to determine forces on polarized bodies except in a few simple cases.

For example, suppose that a parallel-plate capacitor is partially dipped in a liquid dielectric, as in Fig. 9.3a. If the capacitor is charged, polarization charges exist only on the two vertical sides of the dielectric inside the capacitor. The electric force acting on them has only a horizontal component, if any. Yet experiment tells us that when we charge the capacitor, there is a small but noticeable rise in the dielectric level between the plates. How can we explain this phenomenon?

The answer lies in what happens not at the top of the dielectric but near the bottom edge of the capacitor. In that region, the dipoles in the dielectric orient themselves as shown in Fig. 9.3b. The net force on the dipoles points essentially upward and pushes the dielectric up between the plates. Although we can explain the nature of this force, based on what we have learned so far we have no idea how to calculate it. The method described next enables us to determine the electric forces in this and many other cases where the direct method fails. In addition, conceptually the same method is used for the more important determination of magnetic forces in practical applications.

![Diagram of a parallel-plate capacitor partially dipped in a liquid dielectric](image)

**Figure 9.3** (a) When a parallel-plate capacitor dipped in a liquid dielectric is charged, the level between the plates rises due to electric forces acting on dipoles in the dielectric in the region around the edge of the capacitor, where the field is not uniform. (b) Enlarged domain of the capacitor fringing field in the dielectric, indicating the force on a dipole in a nonuniform field.
Figure 9.4 A body in an electrostatic system moved a small distance \( dx \) by the electric force.

Consider an arbitrary electrostatic system consisting of a number of charged conducting and polarized dielectric bodies. We know that there are forces acting on all these bodies. Let us concentrate on one of the bodies, for example the one in Fig. 9.4, that may be either a conductor or a dielectric. Let the unknown electric force on the body be \( \mathbf{F} \), as indicated in the figure.

Suppose we let the electric force move the body by a small distance \( dx \) in the direction of the \( x \) axis indicated in the figure. The electric force would in this case do work equal to

\[
dA_{\text{el.force}} = F_x \, dx,
\]

where \( F_x \) is the projection of the force \( \mathbf{F} \) on the \( x \) axis.

At first glance we seem to have gained nothing by this discussion: we do not know the force \( \mathbf{F} \), so we do not know the work \( dA_{\text{el.force}} \) either. However, we will now show that if we know how the electric energy of the system depends on the coordinate \( x \), we can determine the work \( dA_{\text{el.force}} \), and then from Eq. (9.10), the component \( F_x \) of the force \( \mathbf{F} \). In this process, either (1) the charges on all the bodies of the system can remain unchanged or (2) the potentials of all the conducting bodies can remain unchanged.

Let us consider case (1) first. The charges can remain unchanged in spite of the change in the system geometry only if none of the conducting bodies is connected to a source that could change its charge (for example, a battery). Therefore, by conservation of energy, the work in moving the body can be done only at the expense of the electric energy contained in the system.

Let the system energy as a function of the coordinate \( x \) of the body, \( W_e(x) \), be known. The increment in energy after the displacement, \( dW_e(x) \), is negative because some of the energy has been used for doing the work. Since work has to be a positive number, we have in this case \( dA_{\text{el.force}} = -dW_e(x) \). Combining this expression with Eq. (9.10), the component \( F_x \) of the electric force on the body is

\[
F_x = -\frac{dW_e(x)}{dx} \quad \text{(charges kept constant)}.
\]
Example 9.4—Force acting on one plate of a parallel-plate capacitor. In this example, we will find the electric force acting on one plate of a parallel-plate capacitor. The dielectric is homogeneous, of permittivity $\varepsilon$, the area of the plates is $S$, and the distance between them is $x$. One plate is charged with $Q$ and the other with $-Q$ (Fig. 9.5). Let the electric force move the right plate by a small distance $dx$. The energy in the capacitor is given by $W_e(x) = Q^2/2C(x) = Q^2x/(2\varepsilon S)$, so the force that tends to increase the distance between the plates is

$$F_x = -\frac{dW_e(x)}{dx} = -\frac{Q^2}{2\varepsilon S}.$$

This is the same result as in Example 9.2, except for the sign. The minus sign tells us that the force tends to decrease the coordinate $x$, i.e., that it is attractive.

Example 9.5—Force per unit length acting on a conductor of a two-wire line. The wires of a two-wire line of radii $a$ are $x$ apart, and are charged with charges $Q'$ and $-Q'$. The energy per unit length of the line is

$$W'_e(x) = \frac{Q'^2}{2C'} = \frac{Q'^2}{2\pi \varepsilon_0} \ln \frac{x}{a},$$

using $C'$ as calculated in problem P8.13. From Eq. (9.11) we obtain the force per unit length on the right conductor, tending to increase the distance between them, as

$$F_x = -\frac{dW'_e}{dx} = -\frac{Q'^2}{2\pi \varepsilon_0 x}.$$

This is the same as in Example 9.3, except for the minus sign. We know that this means only that the force tends to decrease the distance $x$ between the wires, i.e., that it is attractive.

Example 9.6—Force acting on a dielectric partly inserted into a parallel-plate capacitor. Let us find the electric force acting on the dielectric in Fig. 9.6. Equation (9.11) allows us to do this in a simple way. The capacitance of a capacitor such as this one is given by
Figure 9.6 Determination of the force on the dielectric partly inserted between the electrodes of a parallel-plate capacitor using Eq. (9.11)

\[ C = C_1 + C_2 = \epsilon \frac{bx}{d} + \epsilon_0 \frac{b(a-x)}{d} \]

(see problem P8.8). The energy in the capacitor is

\[ W_e(x) = \frac{Q^2}{2C} = \frac{Q^2}{2(C_1 + C_2)} = \frac{Q^2d}{2b[\epsilon x + \epsilon_0(a-x)]}. \]

The derivative \(dW_e(x)/dx\) in this case is a bit more complicated to calculate, and it is left as an exercise. The force is found to be

\[ F_x = \frac{V^2 b}{2d} (\epsilon - \epsilon_0). \]

Note that this force is always positive because \(\epsilon > \epsilon_0\). This means that the forces tend to pull the dielectric further in between the plates.

**Example 9.7—Rise of level of liquid dielectric partly filling a parallel-plate capacitor.**

As a final example of the application of Eq. (9.11), let us determine the force that raises the level of the liquid dielectric between the plates of the capacitor in Fig. 9.3. Assume the dielectric is distilled water with \(\epsilon_r = 81\), the width of the plates is \(b\), their distance is \(d = 1\) cm, and the capacitor was charged by being connected to \(V = 1000\) V. The electric forces will raise the level of the water between the plates until the weight of the water between the plates becomes equal to this force. The weight is equal to

\[ G = \rho_m x b d g, \]

where \(\rho_m\) is the mass density of water and \(g = 9.81\) m/s\(^2\). By equating this force to the force that we found in Example 9.6, we get

\[ \rho_m x b d g = \frac{V^2 b}{2d} (\epsilon - \epsilon_0) \]

\[ x = \frac{V^2}{2d^2 \rho_m g} (\epsilon - \epsilon_0) = 1.44\ \text{mm}. \]
So far, we have discussed examples of case (1), where the charges in a system were kept constant. Case (2) is finding forces from energy when the voltage, not the charge, of the \( n \) conducting bodies of the system is kept constant (for example, we connect the system to a battery). When a body is moved by electric forces again by \( dx \), some changes must occur in the charges on the conducting bodies, due to electrostatic induction. These changes are made at the expense of the energy in the sources (battery). So we would expect the energy contained in the electric field to increase in this case. It can be shown in a relatively straightforward way that the expression for the component \( F_x \) of the electric force on the body in this case is

\[
F_x = +\frac{dW_e(x)}{dx} \quad \text{(potentials kept constant).} \tag{9.12}
\]

Of course, this formula in all cases leads to the same result for the force as Eq. (9.11), but in some cases it is easier to calculate \( dW_e/dx \) for constant potentials than for constant charges, and conversely.

**Example 9.8—Example 9.6 revisited.** Let us compute the force from Example 9.6 using Eq. (9.12) instead of Eq. (9.11), which we used in Example 9.6. Now we assume the potential of the two plates to be constant, and therefore express the system energy in the form

\[
W_e(x) = \frac{1}{2} CV^2 = \frac{V^2}{2} \left[ \frac{bx}{\epsilon} + \epsilon_0 \frac{b(a-x)}{d} \right],
\]

so that

\[
F_x = +\frac{dW_e}{dx} = \frac{V^2 b}{2 d} (\epsilon - \epsilon_0).
\]

The result is easier to obtain than in Example 9.6.

**Questions and problems:** P9.17 to P9.20

### 9.6 Electrostatic Pressure on Boundary Surfaces

In an electrostatic field there is pressure on all boundary surfaces. Although it is always small in terms of the pressure values we encounter around us (e.g., pressure of air in tires, pressure on pistons of combustible engines), it has interesting applications. Therefore we will derive the general expression for pressure on the boundary surface between two dielectrics, and estimate its magnitude.

Assume first that the boundary surface is tangential to the lines of the electric field strength vector (Fig. 9.7a). Let the electric forces push the surface by a small distance \( dx \) from dielectric 2 toward dielectric 1, as in the figure. Since the lines of the vector \( \mathbf{E} \) are tangential to the surface, the boundary conditions have not changed, so \( \mathbf{E} \) remains the same. Therefore, the potential difference between any two bodies in the system remains the same as well. This means we have to use Eq. (9.12) to determine the force per unit area on the boundary surface.
Figure 9.7 Boundary surfaces between two dielectrics. (a) The lines of the electric field strength $\mathbf{E}$ are tangential to the boundary. (b) The lines of the electric displacement vector $\mathbf{D}$ are normal to the boundary.

The energy in the system did change, since in the thin layer of thickness $dx$ the energy density before the displacement was $\epsilon_1 E_{\text{tang}}^2/2$, and after the displacement it became $\epsilon_2 E_{\text{tang}}^2/2$. If we consider a small patch of the boundary surface of area $\Delta S$, Eq. (9.12) yields

$$ (F_x)_{\text{on } \Delta S} = +\frac{d}{dx} \left[ \frac{1}{2} \left( \epsilon_2 E_{\text{tang}}^2 - \epsilon_1 E_{\text{tang}}^2 \right) \right] dx \Delta S, $$

(9.13)

from which the pressure on the boundary surface is

$$ p_{\text{E tang}} = \frac{(F_x)_{\text{on } \Delta S}}{\Delta S} = \frac{1}{2} (\epsilon_2 - \epsilon_1) E_{\text{tang}}^2. $$

(9.14)

Note that the pressure acts toward the dielectric of smaller permittivity.

Consider now the case in Fig. 9.7b, where the boundary surface is such that the lines of the electric displacement vector $\mathbf{D}$ are normal to it. Assume again that due to electric forces, the surface is displaced by a small distance $dx$. The boundary conditions for vector $\mathbf{D}$ are satisfied, so it will not change. According to generalized Gauss’ law, the charges on conducting bodies will therefore not be changed either. Hence this case corresponds to the formula in Eq. (9.11).

Again, in this case the energy density changed in the thin layer of thickness $dx$, so the force on a small patch of the boundary surface of area $\Delta S$ is found as

$$ (F_x)_{\text{on } \Delta S} = -\frac{d}{dx} \left[ \frac{1}{2} \left( \frac{D_{\text{norm}}^2}{\epsilon_2} - \frac{D_{\text{norm}}^2}{\epsilon_1} \right) \right] dx \Delta S. $$

(9.15)

The electrostatic pressure in this case is thus

$$ p_{D_{\text{norm}}} = \frac{1}{2} \left( \frac{1}{\epsilon_1} - \frac{1}{\epsilon_2} \right) D_{\text{norm}}^2. $$

(9.16)

Note that in this case also the pressure acts toward the dielectric of smaller permittivity.
The lines of vector $E$ are rarely tangential, and lines of vector $D$ rarely normal, to boundary surfaces. When they are at an arbitrary angle with respect to the surface, the energy density in either of the two dielectrics can be expressed as

$$\frac{1}{2} \epsilon E^2 = \frac{1}{2} (\epsilon E_{\text{tang}}^2 + \epsilon E_{\text{norm}}^2) = \frac{1}{2} (\epsilon E_{\text{tang}}^2 + D_{\text{norm}}^2 / \epsilon).$$  \hspace{1cm} (9.17)

This means that the pressure due to the electrostatic field in the general case is given as the sum of the pressures in Eqs. (9.14) and (9.16):

$$p = \frac{1}{2} (\epsilon_2 - \epsilon_1) \left( E_{\text{tang}}^2 + \frac{D_{\text{norm}}^2}{\epsilon_1 \epsilon_2} \right).$$  \hspace{1cm} (9.18)

It is interesting that from Eq. (9.16) we can also obtain the pressure on the surface of a charged conductor. Let the conductor be medium 2, and assume that $\epsilon_2 \to \infty$, which implies that it is “infinitely polarizable,” an electrostatic equivalent to a conductor. Replacing $\epsilon_1$ by $\epsilon$, Eq. (9.16) yields

$$p_{\text{on conductor surface}} = \frac{1}{2} \frac{D_{\text{norm}}^2}{\epsilon} = \frac{1}{2} E \cdot D.$$  \hspace{1cm} (9.19)

The pressure is directed toward the dielectric.

**Example 9.9**—Pressure on a liquid dielectric between plates of a parallel-plate capacitor. Consider again the parallel-plate capacitor dipped into a liquid dielectric, Fig. 9.3a. Eq. (9.14) tells us immediately that there is an upward pressure on the upper surface of the dielectric. It is left as an exercise for the reader to show that the same result is obtained as before, but in a much simpler way.

**Example 9.10**—Force acting on a plate of a parallel-plate capacitor. The force on one of the plates of the parallel-plate capacitor (from Example 9.4) can now be obtained easily using Eq. (9.19). Note that we know the field on the plate surface if we know either the voltage between the plates or the charge of a plate (assumed to be distributed uniformly over it). The completion of this example is also left to the reader.

**Example 9.11**—Magnitude of electrostatic pressure on a dielectric surface. Let us now do a simple calculation that will tell us how strong electrostatic pressures can be. Imagine a slab of dielectric of $\epsilon_r = 4$ (say, quartz) is placed in an electric field perpendicular to the field lines (Fig. 9.8). Let us find the pressure on the front side of the slab for the strongest possible field in air, $E_0 = 30 \text{kV/cm}$. Using Eq. (9.16), we obtain

$$p_{\text{Dnorm}} = \frac{1}{2} \epsilon_0 \left( 1 - \frac{1}{4} \right) D_0^2 = \frac{3}{8} E_0^2 \epsilon_0 \epsilon_0 \simeq 30 \text{ Pa}.$$  

In comparison, typical pressure inside a car tire is 200 kPa (30 psi), or four orders of magnitude larger. [A pascal (Pa) is the SI unit for pressure equal to N/m². The psi stands for "pounds per square inch."]

**Questions and problems:** Q9.20 to Q9.22, P9.21 and P9.22
9.7 Chapter Summary

1. Electrostatic energy, forces, and pressures are small when compared with the usual magnitude of these quantities around us. However, we will later show that they have considerable practical significance.

2. Electrostatic energy can be considered as a potential energy of a system of charges, or as distributed throughout the field with a density equal to \( \frac{1}{2} \epsilon E^2 \).

3. Electric forces can be obtained directly only if the charge distribution is known, which is rarely the case. Therefore a method for determining the forces based essentially on the law of conservation of energy has been derived. It enables the forces to be found from energy.

4. There is a pressure acting on all boundary surfaces in an electrostatic field. It is always directed toward the medium of lower permittivity.

Questions

Q9.1. What force drives electric charges that form electric current through circuit wires?

Q9.2. Capacitors of capacitances \( C_1, C_2, \ldots, C_n \) are connected (1) in parallel, or (2) in series with a source of voltage \( V \). Determine the energy in the capacitors in both cases.

Q9.3. A parallel-plate capacitor with an air dielectric, plate area \( S \), and distance \( d \) between plates is charged with a fixed charge \( Q \). If the distance between the plates is increased by \( dx \) (\( dx > 0 \)), what is the change in electric energy stored in the capacitor? Explain the result.

Q9.4. Repeat question Q9.3 assuming that \( dx < 0 \).

Q9.5. A parallel-plate capacitor with an air dielectric and capacitance \( C_0 \) is charged with a charge \( Q \). The space between the electrodes is then filled with a liquid dielectric of permittivity \( \epsilon \). Determine the change in the electrostatic energy stored in the capacitor. Explain the result.

Q9.6. Can the density of electric energy be negative? Explain.
Q9.7. If you charge a 1-pF capacitor by connecting it to a source of 100 V, do you think the energy contained in the capacitor can damage a semiconductor device if discharged through it? Explain.

Q9.8. If you touch your two hands to the electrodes of a charged high-voltage capacitor, what do you think are the principal dangers to your body?

Q9.9. Explain in your own words why a polarized dielectric contains energy distributed throughout the dielectric.

Q9.10. Discuss whether a system of charged bodies can have zero total electric energy.

Q9.11. Can the electric energy of a system of charges be negative?

Q9.12. Explain in detail how you would calculate approximately the force $F_{12}$ in Eq. (9.8), assuming that you know the charge distribution on the two bodies in Fig. 9.2.

Q9.13. If the field induces a dipole moment in a small body, it will also tend to move the body toward the region of stronger field. Sketch an inhomogeneous field and the dipole, and explain.

Q9.14. Under the influence of electric forces in a system, a body is rotated by a small angle. The system consists of charged, insulated conducting bodies. Is the energy of the system after the rotation the same as before, larger than before, or smaller than before? Explain.

Q9.15. If we say that $dW_r$ is negative, what does this mean?

Q9.16. Is weight a force? If it is, what kind of force? If it is not, what else might it be?

Q9.17. Is it possible to have a system of three point charges that are in equilibrium under the influence of their own mutual electric forces? If you can find such a system, is the equilibrium stable or unstable?

Q9.18. A soap bubble can be viewed as a small stretchable conducting ball. If charged, will it stretch or shrink? Do you think the change in size can be observed?

Q9.19. Explain why a charged body attracts uncharged small bodies of any kind.

Q9.20. Explain why, in Eq. (9.13), we subtracted the energy density in the first medium from the energy density in the second medium, and not the other way around.

Q9.21. A glass of water is introduced into an arbitrary inhomogeneous electric field. What is the direction of the pressure on the water surface?

Q9.22. Derive Eq. (9.19) from Eq. (9.18).

PROBLEMS

P9.1. A bullet of mass 10 g is fired with a velocity of 800 m/s. How many high-voltage capacitors of capacitance 1 μF can you charge to a voltage of 10 kV with the energy of the bullet?

P9.2. A coaxial cable 1 long, of inner radius $a$ and outer radius $b$, is first filled with a liquid dielectric of permittivity $\epsilon$. Then it is connected for a short time to a battery of voltage $V$. After the battery is disconnected, the dielectric is drained out of the cable. (1) Find the voltage between the cable conductors after the dielectric is drained out of the cable. (2) Find the energy in the cable before and after the dielectric is drained.

P9.3. A spherical capacitor with an air dielectric, of electrode radii $a = 10$ cm and $b = 20$ cm, is charged with a maximum charge for which there is still no air breakdown around the inner electrode of the capacitor. Determine the electric energy of the system.
P9.4. Repeat the preceding problem for a coaxial cable of length \( d = 10 \text{ km} \), of conductor radii \( a = 0.5 \text{ cm} \) and \( b = 1.2 \text{ cm} \).

P9.5. Calculate the largest possible electric energy density in air. How does this energy density compare with a 0.5 J/cm\(^3\) chemical energy density of a mixture of some fuel and compressed air?

P9.6. Show that half of the energy inside a coaxial cable with a homogeneous dielectric, of inner conductor radius \( a \) and outer conductor radius \( b \), is contained inside a cylinder of radius \( a < r < \sqrt{ab} \).

P9.7. A metal ball of radius \( a = 10 \text{ cm} \) is placed in distilled water (\( \epsilon_r = 81 \)) and charged with \( Q = 10^{-9} \text{ C} \). Find the energy that was used up to charge the ball.

P9.8. A dielectric sphere of radius \( a \) and permittivity \( \epsilon \) is situated in a vacuum and is charged throughout its volume with volume density of free charges \( \rho(r) = \rho_0 a/r \), where \( r \) is the distance from the sphere center. Determine the electric energy of the sphere.

P9.9. Repeat the preceding problem if the volume density of free charges is constant, equal to \( \rho \).

P9.10. Inside a hollow metal sphere, of inner radius \( b \) and outer radius \( c \), is a metal sphere of radius \( a \). The centers of the two spheres coincide (concentric spheres), and the dielectric is air. If the inner sphere carries a charge \( Q_1 \) and the outer sphere a charge \( Q_2 \), what is the energy stored in the system?

*P9.11. Prove Thomson’s theorem: the distribution of static charges on conductors is such that the energy of the system of charged conductors is minimal.

*P9.12. Prove that if an uncharged conductor, or a conductor at zero potential, is introduced into an electrostatic field produced by charges distributed on conducting bodies, the energy of the system decreases.

P9.13. An electric dipole of moment \( \mathbf{p} \) is situated in a uniform electric field \( \mathbf{E} \). If the angle between the vectors \( \mathbf{p} \) and \( \mathbf{E} \) is \( \alpha \), find the torque of the electric forces acting on the dipole. What do the electric forces tend to do?

P9.14. An electric dipole of moment \( \mathbf{p} = Qd \) is situated in an electric field of a negative point charge \( Q_0 \), at a distance \( r \gg d \) from the point charge. If the vector \( \mathbf{p} \) is oriented toward the point charge, find the total electric force acting on the dipole.

P9.15. A two-wire line has conductors with radii \( a = 3 \text{ mm} \) and the wires are \( d = 30 \text{ cm} \) apart. The wires are connected to a voltage generator such that the voltage between them is on the verge of initiating air ionization. (1) Find the electric energy per unit length of this line. (2) Find the force per unit length acting on each of the line wires.

P9.16. A conducting sphere of radius \( a \) is cut into two halves, which are pressed together by a spring inside the sphere. The sphere is situated in air and is charged with a charge \( Q \). Determine the force on the spring due to the charge on the sphere. In particular, if \( a = 10 \text{ cm} \), determine the force corresponding to the maximal charge of the sphere in air for which there is no air breakdown on the sphere surface.

P9.17. Find the electric force acting on the dielectrics labeled 1 and 2 in the parallel-plate capacitor in Fig. P9.17. The capacitor plates are charged with \( Q \) and \(-Q\). Neglect edge effects.

P9.18. The inner conductor of the coaxial cable in Fig. P9.18 can slide along the cylindrical hole inside the dielectric filling. If the cable is connected to a voltage \( V \), find the electric force acting on the inner conductor.

P9.19. One end of an air-filled coaxial cable with inner radius \( a = 1.2 \text{ mm} \) and an outer radius of \( b = 1.5 \text{ mm} \) is dipped into a liquid dielectric. The dielectric has a density of mass equal to \( \rho_m = 0.8 \text{ g/cm}^3 \), and an unknown permittivity. The cable is connected to a voltage
$V = 1000$ V. Due to electric forces, the level of liquid dielectric in the cable is $h = 3.29$ cm higher than the level outside the cable. Find the approximate relative permittivity of the liquid dielectric, assuming the surface of the liquid in the cable is flat.

P9.20. The end of a coaxial cable is closed by a dielectric piston of permittivity $\varepsilon$ and length $x$. The radii of the cable conductors are $a$ and $b$, and the dielectric in the other part of the cable is air. What is the magnitude and direction of the axial force acting on the dielectric piston, if the potential difference between the conductors is $V$?

P9.21. One branch of a U-shaped dielectric tube filled with a liquid dielectric of unknown permittivity is situated between the plates of a parallel-plate capacitor (Fig. P9.21). The voltage between the capacitor plates is $V$, and the distance between them $d$. The cross section of the U-tube is a very thin rectangle, with the larger side parallel to the electric field intensity vector in the charged capacitor. The dielectric in the tube above the liquid dielectric is air, and the mass density of the liquid dielectric is $\rho_m$. Assume that $h$ is the measured difference between the levels of the liquid dielectric in the two branches of the U-tube. Determine the permittivity of the dielectric.

P9.22. A soap bubble of radius $R = 2$ cm is charged with the maximal charge for which breakdown of air on its surface does not occur. Calculate the electrostatic pressure on the bubble.