Volume diffraction

Statement of problem

What if the material isn’t homogeneous? In the weak perturbation limit, we can find a closed-form solution.

We will define that the dielectric tensor is made up of a homogenous term (unperturbed material) and an inhomogeneous term (where all of our fun physics will come in):

\[
\mathbf{\varepsilon}(t, x, y, z) = \varepsilon_0 \left[ \mathbf{\varepsilon}_H(t) + \mathbf{\varepsilon}_{IH}(t, x, y, z) \right]
\]

The perturbation should have little or no DC content (that should be in the homogeneous term).
### K-space

**Derivation**

To solve this perturbed problem, first solve for the eigenfunctions of the unperturbed (vector version):

\[
\vec{E}(t,x,y,z) = \mathcal{F}_{txy}^{-1} \left\{ \sum_p \vec{E}(\omega, k_x, k_y, z = 0) \cdot \vec{e}_p e^{-jk_{p,z}(\omega k_x, k_y)z} \right\}
\]

Now write the perturbed wave equation including the inhomogeneous dielectric:

\[
\vec{D} = \left( \varepsilon_0 \varepsilon_H \ast \vec{E} \right) + \vec{P}_{IH} \quad \vec{P}_{IH}(t, \vec{r}) = \varepsilon_0 \varepsilon_{IH}(t, \vec{r}) \ast \vec{E}(t, \vec{r}) \\
\vec{P}_{IH}(\omega, \vec{k}) = \varepsilon_0 \varepsilon_{IH}(\omega, \vec{K}) \ast \vec{E}(\omega, \vec{k})
\]

\[
\nabla \times \nabla \times \vec{E} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \varepsilon_H \ast \vec{E} = -\mu_0 \frac{\partial^2}{\partial t^2} \vec{P}_{IH}(t, x, y, z)
\]

Use the method of “undetermined coefficients” in which the amplitudes of the unperturbed eigenfunctions are allowed to vary with distance:

\[
\mathcal{E}_p(\omega, k_x, k_y, z = 0) = \mathcal{F}_{txy} \left[ \vec{E}(t, x, y, 0) \right] \cdot \vec{e}_p (\omega, k_x, k_y)
\]

\[
\mathcal{E}_p(t, x, y, z) = \mathcal{F}_{txy}^{-1} \left\{ \mathcal{E}_p(\omega, k_x, k_y, z) e^{-jk_{p,z}(\omega k_x, k_y)z} \right\}
\]

If there is no perturbation \( \varepsilon_{IH}=0 \), then we expect that \( \mathcal{E}_p \) will have no z dependence and thus the solution will return to the homogeneous, Fourier propagation result.

The field in real space is now:

\[
\vec{E}(t,x,y,z) = \mathcal{F}_{txy}^{-1} \left\{ \sum_p \mathcal{E}_p(\omega, k_x, k_y, z) \hat{e}_p e^{-jk_{p,z}(\omega k_x, k_y)z} \right\}
\]
**Derivation of the DE**

Continued

Assume perturbation sufficiently weak that can ignore first term:

\[
\nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E} = -\mu_0 \frac{\partial^2}{\partial t^2} \vec{P}_{IH}(t, x, y, z)
\]

Substitute in assumed solution:

\[
\vec{E}(t, x, y, z) = F_{txy}^{-1} \left\{ \sum_p \mathbf{E}_p(\omega, k_x, k_y, z, z) \hat{e}_p e^{-jk_{p,z}(\omega, k_x, k_z)z} \right\}
\]

Because the Fourier basis is orthogonal, one can multiply by \(e^{-j(\omega-t)}\) and integrate over \(\omega, k_x, k_y\), collapsing the FT integral on the LHS to a delta function and taking the transverse FT of the RHS. To remove the sum over polarizations, we use vector inner products. Unfortunately, the \(e_p\) are not orthogonal, so once again we must construct a basis set, this time in 3 dimensions (not just two as used on the boundary):

\[
k^2_{p,z} \mathbf{E}_p + 2jk_{p,z} \frac{d}{dz} \mathbf{E}_p - \frac{\omega^2}{c^2} \vec{e}_p \vec{E}_H(\omega) \hat{e}_p \mathbf{E}_p = \mu_0 \omega^2 \vec{P}_{IH}(\omega, k_x, k_z, z) \cdot \vec{e}_p e^{jk_{p,z}z}
\]

where \(\vec{e}_l \cdot \vec{e}_m = \delta_{lm}\). When the polarizations are orthogonal, \(\vec{e}_l = \hat{e}_l\).

Cancel out terms which are unperturbed solution (this is why we used it):

\[
-\frac{d^2}{dz^2} \mathbf{E}_p + 2jk_{p,z} \frac{d}{dz} \mathbf{E}_p = \mu_0 \omega^2 \vec{P}_{IH}(\omega, k_x, k_z, z) \cdot \vec{e}_p e^{jk_{p,z}z}
\]

Assume variation of envelope is slow in comparison to \(\lambda\). This is the “slowly varying envelope approximation.” It discards backwards travelling waves.

\[
2jk_{p,z} \frac{d}{dz} \mathbf{E}_p = \frac{k^2}{\varepsilon_0} \vec{P}_{IH}(\omega, k_x, k_z, z) \cdot \vec{e}_p e^{jk_{p,z}z}
\]
Derivation of k-space

First-order Born approximation

Integrate remaining first-order ordinary differential equation:

\[ \mathbf{E}_p(\omega, k_x, k_y, z) = -j \int_0^z \frac{k_0^2}{2\varepsilon_0 k_{p,z}} \mathcal{F}_{lxy} [\bar{P}_{IH}(t, x, y, z')] \cdot \tilde{e}_p(\omega, k_x, k_y) e^{jk_z z'} dz' \]

Assume perturbation is confined to region \( z = 0 \) to \( L \) and thus for \( z>L \), form of integral is Fourier transform:

\[ \mathbf{E}_p(\omega, k_x, k_y, L) = -j \frac{k_0^2}{2\varepsilon_0 k_{p,z}} \tilde{e}_p \cdot \mathcal{F}_{lxy} [\bar{P}_{IH}(t, x, y, z')]|_{k_z'=k_{p,z}(\omega,k_x,k_y)} \]

and substitute the known inhomogeneous polarization field:

\[ = -j \frac{k_0^2}{2k_{p,z}} \tilde{e}_p \cdot [\overline{\mathcal{E}}_{IH}(\omega, \vec{K}) \ast_{\vec{k}} \bar{E}(\omega, \vec{k})]|_{k_z'=k_{p,z}(\omega,k_x,k_y)} \]

• This is a transcendental equation for \( E \).
• If the diffraction efficiency is low, then \( E \) on the right-hand side can be approximated as the incident field.
• This is formally the first term in a perturbation series.
• A.k.a. first-order Born approximation, undepleted-pump approximation, single-scattering limit

\[ \mathbf{E}_p^i(\omega, k_x, k_y, L) = -j \frac{k_0^2}{2k_{p,z}} \tilde{e}_p \cdot [\overline{\mathcal{E}}_{IH}(\omega, \vec{K}) \ast_{\vec{k}} \bar{E}^0(\omega, \vec{k})]|_{k_z'=k_{p,z}(\omega,k_x,k_y)} \]
K-space

Graphical tool for volume holography

\[ \mathcal{E}_{p}^{1}(\omega, k_{x}, k_{y}, L) = -j \frac{k_{0}^{2}}{2k_{p,z}} \tilde{E}_{p} \cdot \left[ \mathcal{E}_{	ext{IH}}(\omega, \vec{K}) \ast \vec{E}^{0}(\omega, \vec{k}) \right]_{k_{z}=k_{p,z}(\omega, k_{x}, k_{y})} \]

1. Given the vector boundary field \( \vec{E}^{0}(t, x, y, z = 0) \)
   a. Fourier transform each vector component to find \( \vec{E}^{0}(\omega, k_{x}, k_{y}, z = 0) \)
   b. We then know the incident field in Fourier space:
      \[ \vec{E}^{0}(\omega, k_{x}, k_{y}, k_{z}) = \sum_{p} \hat{e}_{p} \left[ \mathcal{E}_{p} \cdot \vec{E}^{0}(\omega, k_{x}, k_{y}, 0) \right]_{k_{z}=k_{p,z}(\omega, k_{x}, k_{y})} \]

2. Given \( \mathcal{E}_{	ext{IH}}(t, x, y, z) \), Fourier transform to find \( \mathcal{E}_{	ext{IH}}(\omega, k_{x}, k_{y}, k_{z}) \)
3. \( P_{	ext{IH}}(\omega, k_{x}, k_{y}, k_{z}) \) is the convolution of 1 and 2.
4. Sample 3 on the allowed propagating mode to find \( \mathcal{E}(\omega, k_{x}, k_{y}, z=L) \)

\[ k_{0}^{x} + K_{x} = k_{x}^{1} \]
\[ \frac{2\pi}{\lambda_{0}} \sin \theta^{0} + \frac{2\pi}{\Lambda} = \frac{2\pi}{\lambda_{0}} \sin \theta^{1} \]
\[ -\sin \theta^{0} = \sin \theta^{1} = \frac{\Lambda}{2\lambda_{0}} \]

Isotropic Bragg matching.

Real space

\[ A \]

Incident \[ z \]

Diffracted

Grating period \( \Lambda \)

\[ x \]

\[ \rightarrow \]

\[ L \]

\[ \rightarrow \]

\[ k_{x} \]

\[ k_{y} \]

\[ k_{z} \]

\[ \frac{4\pi}{L} \]

\[ \frac{4\pi}{A} \]

\[ \frac{2\pi}{A} \]
Example

Acoustooptic diffraction

- K-space
- Applications
AO diffraction from diamond-shaped transducer

K-space
- Applications
K-space equation

Volume hologram efficiency (weak limit)

\( \mathcal{E}_p^1(\omega, k_x, k_y, L) = -j \frac{k_0^2}{2k_{p,z}} \tilde{e}_p \cdot \left[ \tilde{\mathcal{E}}_{IH}(\omega, \vec{K}) \ast \vec{E}_0(\omega, \vec{k}) \right] \bigg|_{k_z = k_{p,z}(\omega, k_x, k_y)} \)

- If perturbation written by two-wave mixing that creates a grating in-phase with the interference, the diffraction is \( \pi/2 \) out of phase.

- The diffraction efficiency of a uniform grating of thickness \( L \) and strength \( \mathcal{E}_{IH} = n_o \delta n \) is

\[
\eta = \frac{|\mathcal{E}^1(z = L)|^2}{|E^0(\omega, k_x, k_y, 0)|^2} = \frac{j k_0^2}{2k_z} L \mathcal{E}_{IH} = \left( \frac{(2\pi/\lambda_0)^2}{2(2\pi n_0 \cos \theta^1/\lambda_0)} \right)^2 L (n_o \delta n) = \left( \frac{\pi \delta n}{\cos \theta^1 \lambda_0} \right)^2 L = [\kappa L]^2
\]

Koglenik strong perturbation solution:

\[
\eta = \sin^2 \left( \frac{\pi L \delta n}{\cos \theta^1 \lambda_0} \right)
\]

which agrees for weak diffraction.

Translation from \( \Delta \varepsilon \) to \( \Delta n \)

\[
n^2 = [n_H + \delta n \cos(\vec{K} \cdot \vec{r})]^2
\approx n_H^2 + 2n_0 \delta n \cos(\vec{K} \cdot \vec{r}) = \mathcal{E}_H + \mathcal{E}_{IH} \left[ e^{i\vec{K} \cdot \vec{r}} + e^{-i\vec{K} \cdot \vec{r}} \right] = \mathcal{E}_H + 2\mathcal{E}_{IH} \cos(\vec{K} \cdot \vec{r}) \;
\therefore \mathcal{E}_{IH} = n_0 \delta n
\]

Koglenik defines intensity normal to \( z \):

\[
\eta = \frac{I_{Diff} \cdot \hat{z}}{I_{Inc} \cdot \hat{z}} = \frac{|\mathcal{E}^1(z = L)|^2 \cos \theta^1}{|E^0(\omega, k_x, k_y, 0)|^2 \cos \theta^0} = \left[ \frac{\pi L \delta n}{\sqrt{\cos \theta^1 \cos \theta^0 \lambda_0}} \right]^2
\]

It's easy to lose factors of 2 here.

Robert R. McLeod, University of Colorado
K-space equation

Volume hologram efficiency
(strong limit, Bragg matched)

If the two waves are perfectly Bragg (phase) matched, each scattering efficiency $\mathcal{E}^n$ is given by the $z$ integral of the envelope of the source $\mathcal{E}^{n-1}$.

$$\mathcal{E}^1(z) = -j \mathcal{E}^0 \kappa z$$

$$\mathcal{E}^2(z) = -j \kappa \int_0^z \mathcal{E}^1(z') dz' = -\mathcal{E}^0 \frac{1}{2} (\kappa z)^2$$

$$\mathcal{E}^3(z) = -j \kappa \int_0^z \mathcal{E}^2(z') dz' = -j \mathcal{E}^0 \frac{1}{6} (\kappa z)^3$$

Summing all terms at each of the two propagation directions:

$$\mathcal{E}^{\text{diff}}(z) = \sum_n \mathcal{E}^{2n+1}(z) = -j \sin \left( \frac{\pi \Delta n}{\cos \theta' \lambda_0} z \right)$$

$$\mathcal{E}^{\text{inc}}(z) = \sum_n \mathcal{E}^{2n}(z) = \cos \left( \frac{\pi \Delta n}{\cos \theta' \lambda_0} z \right)$$

Which is the Kogelnik solution. So the method can be extended to multiple scattering and high efficiency via continuing the perturbation series, at least in the perfectly Bragg-matched case. What about non-Bragg matched?
K-space equation

Volume hologram efficiency
(strong limit, not Bragg matched, 1/2)

If the two waves are not perfectly Bragg matched, each scattering efficiency $\mathcal{E}^n$ is given by the $z$ integral of the envelope of the source $\mathcal{E}^{n-1}$ times a phase shift term:

Not phase matched, so oscillates with small amplitude.

$$
\mathcal{E}^1(z) = -j \kappa \int_0^z \mathcal{E}^0(z') e^{-j\Delta k_z z'} dz' = \frac{\kappa}{\Delta k_z} (e^{-j\Delta k_z z} - 1) \mathcal{E}^0 = -2j \frac{\kappa}{\Delta k_z} e^{-\frac{z}{2} j\Delta k_z z} \sin\left(\frac{1}{2} \Delta k_z z\right) \mathcal{E}^0
$$

$$
\mathcal{E}^2(z) = -j \kappa \int_0^z \mathcal{E}^1(z') e^{+j\Delta k_z z'} dz' = \left(\frac{\kappa}{\Delta k_z}\right)^2 \left(e^{j\Delta k_z z} - 1 - j\Delta k_z z\right) \mathcal{E}^0
$$

$$
\mathcal{E}^3(z) = -j \kappa \int_0^z \mathcal{E}^2(z') e^{-j\Delta k_z z'} dz' = \left(\frac{\kappa}{\Delta k_z}\right)^3 \left([-2 - j\Delta k_z z] e^{-j\Delta k_z z} + (2 - j\Delta k_z z)\right) \mathcal{E}^0
$$

These collapse to the previous Bragg matched case in the limit of zero phase mismatch. The incident and diffracted fields are the sums of the even and odd terms:

$$
\mathcal{E}^{inc}(z) = \sum_{n \text{ even}} \mathcal{E}^n(z)
$$

$$
\mathcal{E}^{diff}(z) = \sum_{n \text{ odd}} \mathcal{E}^n(z)
$$
K-space equation

Volume hologram efficiency

(strong limit, not Bragg matched, 2/2)

Making the definitions $g^2 \equiv \kappa^2 + \left( \frac{k_{0,z} - k_{1,z}}{2} \right)^2$ and $\gamma \equiv \frac{(k_{0,z} - k_{1,z})/2}{\kappa}$ and expanding the incident and diffracted field envelopes in a power series in $z$:

\[ \mathcal{E}^{\text{inc}}(z) = \sum_{n \text{ even}} \mathcal{E}^n(z) = -j \frac{\kappa}{g} e^{-j\Delta k_z z/2} \left[ g^2 - \frac{1}{6} (g z)^3 + \frac{1}{120} (g z)^5 - \frac{1}{5040} (g z)^7 + \ldots \right] \]

\[ \mathcal{E}^{\text{diff}}(z) = \sum_{n \text{ odd}} \mathcal{E}^n(z) = e^{-j\Delta k_z z/2} \left\{ \left[ 1 - \frac{1}{2} (g z)^2 + \frac{1}{24} (g z)^4 - \frac{1}{720} (g z)^6 + \ldots \right] \right\} \]

The terms in brackets can be recognized as the Taylor series for sine and cosine:

\[ \mathcal{E}^{\text{inc}}(z) = \sum_{n \text{ even}} \mathcal{E}^n(z) = -j \frac{\kappa}{g} e^{-j\Delta k_z z/2} \sin(g z) \]

\[ \mathcal{E}^{\text{diff}}(z) = \sum_{n \text{ odd}} \mathcal{E}^n(z) = e^{-j\Delta k_z z/2} \left[ \cos(g z) - j \frac{\Delta k_z}{2g} \sin(g z) \right] \]

This is the coupled mode solution of the same problem, demonstrating that the perturbation-series approach and the coupled mode approach are the same.
Holographic data storage example
Real space

Record interference between Fourier transform of SLM and a plane-wave reference

Read hologram to reconstruct the data which then is inverse transformed onto a CCD
Holographic data storage example

K-space (recording)

**Recording one page**

Write three gratings (contours show $\xi(\omega, \mathbf{k})$) with one reference wave of width $L$ and three pixel waves of width $A$.

**Recording a second page**

Write rotate the reference and write three new gratings with the same three pixel waves.
Holographic data storage example

Grating space

Recorded gratings (2 pages)

Translate all gratings to origin.
This is \( \varepsilon(\omega k) \)

Utilized grating space

Address space for separation at first null

\[
N_{\text{bits}} = \frac{2k_0 N_{\text{A ref}} (2k_0 N_{\text{obj}})^2}{\left(\frac{2\pi}{L}\right)\left(\frac{2\pi}{A}\right)^2} = \left(2N_{\text{A ref}}/2N_{\text{obj}}\right)^2 \frac{LA^2}{\lambda^3}
\]

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Holographic data storage example

K-space (reading)

Reading one page

Reading the second page
As reference rotates in $\phi$, object Bragg mismatches quickly.

In contrast, as reference rotates in $\theta$, to first-order object stays Bragg-matched and tilts by same angle $\theta$. 
Angular vs. wavelength selectivity for reflection and transmission

Reflection hologram

Wavelength selectivity

Write in blue, shift towards red until reach first null of hologram

Angular selectivity

Write in blue, tilt until reach first null. Note now second-order effect.

Transmission hologram

Wavelength selectivity

Write in blue, shift towards red until reach first null of hologram

Angular selectivity

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Coherent transfer function and impulse response of imaging systems

Consider a transmission microscope with a uniformly filled NA imaging an arbitrary object. What can the microscope actually resolve?

$E_{\text{inc}}(\vec{k}), E_{\text{dif}}(\vec{k})$

$CTF(\vec{K})$

$PSF = \mathcal{F}^{-1}[CTF(\vec{K})]$
Tomography

X-ray projection tomography

“Tomography” = Tomo (slice) + graph (write or draw)

Real (measurement) space

Computed space

Repeat for many angles…
Diffraction tomography

Concept

\[ \mathcal{E}^i(k_x, L) = -j \frac{k_0^2}{2\sqrt{k^2 - k_x^2}} \iint \mathcal{E}_{IH}(\vec{r}) \mathcal{E}^0(\vec{r}) e^{i\left(k_{x,x}\sqrt{k^2 - k_x^2} + k_{z,z}z\right)} d\vec{r} \]

\[ = -j \frac{k_0^2}{2\sqrt{k^2 - k_x^2}} A_0 \iint \mathcal{E}_{IH}(\vec{r}) e^{i\left[k_{x,x}\sqrt{k^2 - k_x^2} + k_{z,z}z - k_{inc}\cdot\vec{r}\right]} d\vec{r} \]

\[ = -j \frac{k_0^2}{2\sqrt{k^2 - k_x^2}} A_0 \mathcal{E}_{IH}(\vec{k}_{Diff} - \vec{k}_{Inc}) \]

Born scattering

yields a single Fourier component of \( \varepsilon \) whose spatial frequency is the difference of diffracted and incident frequencies.

If the incident field is a plane wave \( A_0 e^{-j\vec{k}_{inc}\cdot\vec{r}} \) then we see that \( \mathcal{E}^i(k_x, z) \) is the amplitude of the Fourier component \( \vec{k}_{Diff} - \vec{k}_{Inc} \) of the FT of the object.

If, for example, the incident wave is propagating in \( z \), the diffracted fields are proportional to an arc of the FT of the object given by \( k_x, \sqrt{k^2 - k_x^2} - k \)

Thus to get information about the entire FT, we must use a number of different incident angles to “spin” this arc through the FT.
Diffraction tomography
Implementation

- Tomography
  - Concept

Object under test

Collimated Laser

Measure $E(k_x)$

Rotate sample

$\varepsilon_{IH}(\vec{K})$

Sample rotation

$k_x (1/\mu m)$

$k_z (1/\mu m)$
Forward scattering
1: Fourier transform object

\[ \varepsilon_{IH}(x,z) \]

\[ |\varepsilon_{IH}(K_x,K_z)| \]

Band-limited \( \varepsilon_{IH}(x,z) \)
Forward scattering

2: Diffracted field is arc through $\varepsilon(k)$
Forward scattering

3: Calculate diffraction at each inc angle
Tomographic reconstruction

1: Inverse transform one arc

\[ \text{Re}[E^\dagger(k_t)] \]

\( k_t [1/\mu m] \)

\( E [V/m] \)

Reconstructed Real( \( r_{\text{Re}}(x,y) \))

\( y (\mu m) \)

\( x (\mu m) \)
Tomographic reconstruction

2: Inverse transform another
Tomographic reconstruction

2: Sum them all up

Band-limited object

Re[$\varepsilon(x,z)$]

Im[$\varepsilon(x,z)$]

Reconstruction

Re[$\varepsilon(x,z)$]

Im[$\varepsilon(x,z)$]
Tomography

- Concept

Rotating object and capturing complete forward and backwards scattering

\[ \Delta k = 2 \pi n k_0 \]

or

\[ NA = 2 \pi n \]

or

\[ r_0 = 0.6 \lambda / 2 \pi n \]

= 73 nm

For \( \lambda = 365 \text{ nm} \)

\( n = 1.5 \)
Projection as special case of diffraction tomography

Let the wavelength \(<<\) object  
Diffraction is *straight* slice of FT

1. By the projection-slice theorem, a slice through the origin of the FT of an object has an inverse FT which is the projection (average value) of the object normal to plane of the slice:

\[
\int_{-\infty}^{\infty} f(x, z) \, dz \leftrightarrow F(k_x, 0)
\]

2. The backwards-propagated modes have very nearly the same \(k_z\) (in other words, the \(k\)-surface is nearly flat over the FT of the object). Thus the combined field will suffer very little diffraction or, in other words, the back propagation reduces to a back projection of, in this case, the projected (averaged) object normal to \(k_{inc}\).

3. If the object is amplitude (loss) only, this yields x-ray back projection tomography.
K-space (forward) & ODT (inverse) algorithms

FORWARD

- Calculate the Fourier transform of the inhomogeneous dielectric.
- Choose a set of incidence angles and an array $k_t$ which gives the diffracted angles (relative to the incident) via
  \[ \tan^{-1}\left[ \frac{k_t}{\sqrt{k^2 - k_t^2}} \right] \]

3. For each incident angle, calculate the scattered electric field which is proportional to $\varepsilon_{IH}(K_x,K_y)$ where $K_x,K_y$ are determined from the $\theta_{inc}$ and $k_t$

INVERSE

1. Create an empty array $\varepsilon_{IH-EST}(x,y)$
2. For each incident and diffracted angle ($k_t$), add a complex exponential to the array, weighted by the scattered electric field amplitude
Coordinate rotation
For a single incident angle

How to calculate diffraction vs $k_t$ at various incident angles $\theta_{inc}$

\[
\begin{bmatrix}
K_x \\
K_y
\end{bmatrix} =
\begin{bmatrix}
\cos \theta_{inc} & \sin \theta_{inc} \\
-\sin \theta_{inc} & \cos \theta_{inc}
\end{bmatrix}
\begin{bmatrix}
k_t \\
k_n
\end{bmatrix}
\]

\[
\begin{bmatrix}
K_x \\
K_y
\end{bmatrix} =
\begin{bmatrix}
\cos \theta_{inc} & \sin \theta_{inc} \\
-\sin \theta_{inc} & \cos \theta_{inc}
\end{bmatrix}
\frac{k_t}{\sqrt{k^2 - k_t^2}}
\]

\[
\mathcal{E}^\| (k_t, L) = -j \frac{k_0^2}{2\sqrt{k^2 - k_t^2}} A_0 \mathcal{E}_{iiH}(K_x, K_y)
\]

...and repeat for multiple incident angles $\theta_{inc}$.

Grating space showing sampling of the dielectric at every incidence angle. 50 diffracted fields were recorded at each incidence angle, equally spaced from -0.95 $k$ to 0.95 $k$. 
Irregular inverse DFT

The process of back propagation tomography is an inverse Fourier transform with an irregularly sampled set of points in the $k_x, k_y$ plane. We can deal with this irregular sampling as long as we have fine enough sampling that the function $\varepsilon_{\text{IH}}(K_x, K_y)$ is well approximated by assuming it to be constant in a patch of area $S_p$ centered on each discrete sample:

$$\varepsilon_{\text{IH}}(x, y) = \int \int \varepsilon_{\text{IH}}(K_x, K_y) e^{-j(k_x x + k_y y)} \, dk_x \, dk_y$$  
Inverse FT integral

$$= \sum_p \int \int_{S_p} \varepsilon_{\text{IH}}(K_x^p, K_y^p) e^{-j(K_x^p x + K_y^p y)} \, dk_x \, dk_y$$  
Discrete samples, $p$

$$\approx \sum_p \varepsilon_{\text{IH}}(K_x^p, K_y^p) e^{-j(K_x^p x + K_y^p y)} S_p$$  
Assume uniform over area $S_p$

The differential patch area depends on the location of the sample. Since it doesn’t depend on rotation, we can find it vs. $k_t$ our unrotated transverse sampling of $k$:

$$\delta k_1 = k \left[ \sin^{-1}\left(\frac{k_t + \delta k_t/2}{k}\right) - \sin^{-1}\left(\frac{k_t - \delta k_t/2}{k}\right) \right]$$

$$\delta k_2 = \sqrt{k_x^2 + k_y^2} \delta \theta_{\text{inc}}$$

$$S_p = \delta k_1 \delta k_2$$

We can now include this differential areas in our discrete inverse Fourier transform. It compensates for the fact that there are extra samples near the origin. In the limit of sampling faster than variations of $\varepsilon(k_x, k_y)$ the DFT should converge to the integral FT.
Impact of sample density
Variable # of incidence angles

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_{\text{inc}}$</td>
<td>$-180^\circ$ to $180^\circ$</td>
</tr>
<tr>
<td>$N_{\text{diff}}$</td>
<td>101 samples</td>
</tr>
<tr>
<td>$NA_{\text{diff}}$</td>
<td>+/-0.95</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>1 $\mu$m</td>
</tr>
<tr>
<td>$n_0$</td>
<td>1.5</td>
</tr>
</tbody>
</table>

All quantities IN material

- **Tomography**
  - Parameter study

- TE, $\Delta \theta_{\text{inc}} = 20^\circ$
- TE, $\Delta \theta_{\text{inc}} = 10^\circ$
- TE, $\Delta \theta_{\text{inc}} = 5^\circ$

- TM, $\Delta \theta_{\text{inc}} = 10^\circ$
- TM, $\Delta \theta_{\text{inc}} = 5^\circ$

Images showing reconstructed real parts of $c_{pq}(x,y)$.
Impact of incident NA
Assymetrical transfer function

All quantities IN material

$\Delta \theta_{\text{inc}}$ = $4^\circ$
$N_{\text{diff}}$ = 101 samples
$\text{NA}_{\text{diff}}$ = +/-0.95
$\lambda$ = 1 $\mu$m
$n_0$ = 1.5
Impact of incident NA
Reduced diffracted NA capture

All quantities IN material

- $\Delta \theta_{\text{inc}}$: 4°
- $N_{\text{diff}}$: 101 samples
- $\text{NA}_{\text{diff}}$: +/-0.5
- $\lambda$: 1 μm
- $n_0$: 1.5
Experimental results (1/2)

Write features in a photopolymer with a translating Gaussian beam:

Reconstruct index including magnitude via ODT:

Experimental results (2/2)

Verify that imaging system has sufficient resolution:

Verify that object is in Born limit:
Applications of phase retrieval

• Tomography
  – Measured: intensity at a plane \( z \)
  – Also known:
    - Intensity at a second plane \( z \) OR
    - Object constraints
  – Find: profile of scattering object

• Wavefront sensing (e.g. for adaptive optics)
  – Measured: intensity at a plane \( z \)
  – Also known:
    - Intensity at a second plane \( z \) OR
    - Object constraints OR
    - Local tilt measurements OR...
  – Find: phase and amplitude of \( E \) at \( z \)

• Holographic Optical Trapping
  – Desired intensity profile at target plane
  – Find: best sampled, digitized phase-only hologram at SLM

• Computer generated holography
  – Given: intensity mask at a hologram plane
  – Also known: Desired intensity at target plane
  – Find: profile of mask

• Optical en/decryption
  – Given: encrypted intensity measurement
  – Also known: Encrypted object of finite size AND
  – Find: encrypted data

• Debluring of Hubble images
• Stellar interferometry
• ....
Phase retrieval

Introduction

In the frequency domain, we can relate the electric field at $z=0$ to the field at $z=z$ by

$$E(x,y,z) = F_{xy}^{-1} \left\{ F_{xy} [E(x,y,0)] e^{-jk_z(\omega,k_x,k_y)z} \right\}$$

If $x$ and $y$ are sampled on a discrete grid, this reduces to

$$E(x, y, z) = \text{DFT}_{xy}^{-1} \left\{ \text{DFT}_{xy} [E(x, y, 0)] e^{-jk_z(\omega,k_x,k_y)z} \right\}$$

This is a linear relationship and so could be written

$$\overline{E(z)} = h \cdot \overline{E(0)}$$

Alternatively, one could start in the spatial domain

$$E(x, y, z) = h(x, y; z) * E(x, y, 0)$$

$$h(x, y; z) = \frac{jk_0}{2\pi z} e^{-jk_0\frac{x^2+y^2}{2z}}$$

which would yield the same result. Imagine now we measure only the intensity $I = |E|^2$ at each plane and want to find the phase. The matrix equation above has $2N_xN_y$ equations (since it’s complex) and we have $2N_xN_y$ unknowns (the phase at each plane). BUT, we can not simply invert the matrix to solve for our unknowns. This leads to iterative solutions.
Gerchberg-Saxton
For two intensity measurements

Measure intensities at two planes:

\[
g_0(x, y) = \sqrt{I(x, y, 0)} \quad \text{and} \quad f_0(x, y) = \sqrt{I(x, y, L)}
\]

Take the Fourier transform of the field to be retrieved

\[
G(k_x, k_y) = F_{xy}[g_k(x, y)] = |G|e^{j\varphi_G(k_x, k_y)}
\]

Apply the Fourier-domain constraint. In this case, that means that the amplitude of the spatial-frequency spectrum is invariant.

\[
G'(k_x, k_y) = |F|e^{j\varphi_G(\vec{k})}
\]

Inverse transform back to real space

\[
g'(x, y) = F_{xy}^{-1}[G'(k_x, k_y)] = |g'|e^{j\varphi_{k'}(x, y)}
\]

and apply the real-domain constraint that the amplitude of the field equal the square-root of the measured intensity.

\[
g_{k+1}(x, y) = |g_o|e^{j\varphi_{k'}(x, y)}
\]
Gerchberg-Saxton

2 Intensities, sample results

\[ g_0(x) = \text{rect}(x/80 \, \mu\text{m}) e^{j k_0 r^2 / (2 \, 3000 \, \mu\text{m})} \]

1) Uniform phase offset, 2) can be slow to converge.
Maleki-Devaney

For 1 intensity & compact support

- Know one field is nonzero only in region S
- Measure intensity at one other plane:

\[ g_0(x, y) = \begin{cases} 1 & (x, y) \in S(x, y) \\ 0 & (x, y) \notin S(x, y) \end{cases} \]

Forward propagate the estimate of \( g \) to the plane of \( f (z=L) \)

\[ f(x, y, L) = F_{xy}^{-1} \left\{ F_{xy} \left[ g_k(x, y, 0) \right] e^{-jk_z(\omega, k_x, k_y)L} \right\} \equiv |f| e^{j\phi_f} \]

Apply the measured intensity constraint – the amplitude of the field should agree with the measurement.

\[ f'(x, y) = |f_o| e^{j\phi_f(x, y)} \]

Reverse propagate to the plane of \( g (z=0) \)

\[ g(x, y, 0) = F_{xy}^{-1} \left\{ F_{xy} \left[ f'(x, y, L) \right] e^{+jk_z(\omega, k_x, k_y)L} \right\} \equiv |g| e^{j\phi_g} \]

and apply the compact-support constraint – the amplitude of \( g \) is zero outside \( S(x, y) \).

\[ g_{k+1}(x, y) = |g_o| e^{j\phi_g(x, y)} \]
Maleki-Devaney

For 1 intensity & compact support

Sample results

\[ g_0(x) = \text{rect}(x/80 \, \mu\text{m}) e^{jk_0 r^2/(2 \, 3000 \, \mu\text{m})} \]

Order of magnitude faster than standard GS!
Maleki-Devaney
Applied to ODT

Original Structure

Full phase information

No phase information

Compact support
Convergence
Proof of GS convergence

1) Define least-squared errors at step k in Fourier $E_{F,k}$ and object $E_{O,k}$ space.
2) By Parseval’s theorem, relate these to object and Fourier space sums
3) By observation of algorithm, closest point at every x or $k_x$ yields smaller sum
4) Which gives definition of error in opposite domain

$$E_{F,k}^2 = \frac{1}{N^2} \sum_{i=1}^{N} |G_k(i) - G'_k(i)|^2$$
$$E_{O,k}^2 = \sum_{i=1}^{N} |g_{k+1}(i) - g'_k(i)|^2$$

$$\sum_{i=1}^{N} |g_k(i) - g'_k(i)|^2$$
$$\geq \sum_{i=1}^{N} |g_{k+1}(i) - g'_k(i)|^2$$

$$E_{O,k}^2 = E_{F,k+1}^2$$

Therefore: $E_{F,k}^2 \geq E_{O,k}^2 \geq E_{F,k+1}^2$

Error vs. iteration for previous two examples

Gerchberg-Saxton
Maleki-Devaney