Geometrical optics
Design of imaging systems

“Geometrical optics is either very simple, or else it is very complicated”, Richard P. Feynman

What’s it good for?
1. Where is the image?
2. How large is it?
3. How bright is it?
4. What is the image quality?¹

“Useful when size of aperture is > 100 λ”

Why?

The ray equation
Approx. solution of ME when \( n(r) \) is slow

\[
\vec{E}(\vec{r}) = E(\vec{r}) e^{-jk_0S(\vec{r})}
\]
Assume slowly varying amplitude \( E \) and phase \( S \)

\[
k_0S(\vec{r}) = k_x x + k_y y + k_z z
\]
E.g. plane wave

\[
k_0S(\vec{r}) = k_0 \sqrt{x^2 + y^2 + z^2}
\]
E.g. spherical wave

Substitute into isotropic wave equation and retain lowest terms…

\[
|\nabla S(\vec{r})|^2 = n^2(\vec{r})
\]
Ray equation – reduction of Maxwell’s equations.

Countours of \( S(r) \) at multiples of \( 2\pi \)

\[\text{“Ray” = curve \perp to } S(r)\]

\[
S(r) \quad \text{Optical path length} = \int_A^B n(\vec{r}) ds \quad [\text{m}]
\]
The eikonal equation

An equation for evolution of ray trajectory

Take square-root of ray equation,

\[ \nabla S(\vec{r}) = n(\vec{r})\hat{s} = n(\vec{r})\frac{d\vec{r}}{ds} \]

then take derivative in s,

\[ \frac{d}{ds} [\nabla S(\vec{r})] = \frac{d}{ds} \left[ n(\vec{r})\frac{d\vec{r}}{ds} \right] \]

and chain rule for \( \nabla \),

\[ \frac{d}{ds} [\nabla S(\vec{r})] = \frac{d\vec{r}}{ds} \cdot \nabla [\nabla S(\vec{r})] = \left[ \frac{\nabla S(\vec{r})}{n(\vec{r})} \right] \cdot \nabla [\nabla S(\vec{r})] \]

finally apply another \( \nabla \) identity.

\[ = \frac{1}{2n(\vec{r})} \nabla [\nabla S(\vec{r}) \cdot \nabla S(\vec{r})] = \frac{1}{2n(\vec{r})} \nabla [n^2(\vec{r})] = \nabla n(\vec{r}) \]

\[ \frac{d}{ds} \left[ n(\vec{r})\frac{d\vec{r}}{ds} \right] = \nabla n(\vec{r}) \]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s )</td>
<td>Parametric distance along ray</td>
<td>[m]</td>
</tr>
<tr>
<td>( \vec{r}(s) )</td>
<td>Ray trajectory</td>
<td>[m]</td>
</tr>
</tbody>
</table>
GRIN lenses via eikonal
Good example and important lens technology

\[ n(\rho) = n_0 \left[ 1 - \Delta \left( \frac{\rho}{a} \right)^m \right] \]

Power-law radial index distribution

\[ \nabla n(\rho) = \frac{d}{ds} \left[ n(\rho) \frac{d\hat{r}}{ds} \right] \]

Plug into eikonal

\[ \approx \frac{d}{dz} \left[ n(\rho) \frac{d\hat{r}}{dz} \right] = \frac{d}{dz} \left\{ n(\rho) \left[ \frac{d\rho}{dz} \hat{\rho} + \hat{z} \right] \right\} \]

Paraxial approx.

\[ \frac{d}{d\rho} n(\rho) \approx \frac{d}{dz} \left[ n(\rho) \frac{d\rho}{dz} \right] \]

Simplify with known dependencies

\[ \frac{d^2}{dz^2} \rho \approx \frac{1}{n(\rho)} \frac{d}{d\rho} n(\rho) = \frac{1}{n(\rho)} \frac{d}{d\rho} \left\{ n_0 \left[ 1 - \Delta \left( \frac{\rho}{a} \right)^m \right] \right\} \]

\[ = \frac{1}{n(\rho)} \left[ -n_0 \frac{m\Delta}{a} \left( \frac{\rho}{a} \right)^{m-1} \right] \approx -\frac{m\Delta}{a} \left( \frac{\rho}{a} \right)^{m-1} \]

Plug in \( n(\rho) \)

\[ \frac{d^2}{d\rho^2} \rho \approx \frac{-2\Delta}{a^2} \frac{\rho}{a} = -\kappa^2 \rho \]

Special case \( m=2 \)

\[ \rho(z) = \rho_0 \cos(\kappa z) + \rho_0' \sin(\kappa z) \]

Solution for ray trajectory
GRIN lenses via eikonal

Ray traces

“Full pitch” = imaging lens

“Half pitch” = collimating lens

Fractional pitch (<.5) as typically used
Numerical solution of eikonal
For arbitrary index distributions

3D gradient index distribution

XY slice of $\delta n$

XZ slice of $\delta n$

Ray-trace in XZ

• Design of ideal imaging systems with geometrical optics
  – Ray and eikonal equations
Ray trajectory n=constant

Finally!

\[
\frac{d}{ds} \left[ n(\vec{r}) \frac{d\vec{r}}{ds} \right] = \nabla n(\vec{r}) \quad \text{Eikonal}
\]

\[
\frac{d}{ds} \left[ n_0 \frac{d\vec{r}}{ds} \right] = \nabla n_0 = \vec{0} \quad \text{n(r) = n_o, a constant}
\]

\[
\vec{r}(s) = \vec{c} \ s \quad \text{where } \vec{c} \text{ is a constant.}
\]

Thus, after ~40 slides, we have discovered that, in homogenous materials, rays travel in straight lines!
Fermat’s principle

Another (important) form of the eikonal

Optical path length \( S = \int_{A}^{B} n(\vec{r}) ds \)

- Propagation time = \( S / c \)
- Hero of Alexandria: “Light travels in straight lines”
- Fermat: “Light travels the path which takes the minimum time.”
- Correct: “The time of travel is stationary:”

\[ \partial \int_{A}^{B} n(\vec{r}) ds = 0 \]

- Example 1: Concave mirror with radius of curvature < ellipse
- Example 2: Lifeguard problem
- See Born and Wolf for a derivation from Maxwell’s equations

- Why do we care? At an image point, all OPL must be equal.

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2 R. Feynman, *Lectures on Physics*
Radiometry

Review of terminology & inverse square law

\[ \Omega = \frac{A}{R^2} \]

Solid angle \( \Omega \) is area of sphere subtended \( A \) over radius of sphere \( R \).

\[ I = \frac{\phi}{\Omega} \]

Intensity \( I \) of point source is power \( \phi \) emitted into solid angle \( \Omega \).

\[ E = \frac{\phi}{A} \]

Irradiance \( E \) on surface is power \( \phi \) per unit area \( A \).

\[ E = \frac{I}{R^2} \]

Irradiance of surface by a point source is given by intensity \( I \) of point source over distance to surface \( R \).

“Inverse square law”

<table>
<thead>
<tr>
<th>Q</th>
<th>Energy</th>
<th>[J]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi )</td>
<td>Power (flux)</td>
<td>[J/s=W]</td>
</tr>
<tr>
<td>I</td>
<td>Intensity</td>
<td>[W/sr]</td>
</tr>
<tr>
<td>E</td>
<td>Irradiance</td>
<td>[W/m²]</td>
</tr>
<tr>
<td>L</td>
<td>Radiance (photometric “brightness”)</td>
<td>[W/(sr m²)]</td>
</tr>
</tbody>
</table>
Radiometry via rays

Consider a finite source radiating into a cone

\[ E = \frac{\phi}{\delta A} = \frac{\phi}{4\pi R^2} \]

Irradiance \( E \) on surface \( \delta A \) given power \( \phi \) into cone

Now launch a set of rays inside this cone and examine the ray density as a function of radius

\[ \rho \equiv \frac{\delta N}{\delta A} = \frac{\delta N}{4\pi R^2} \propto E \]

A general proof that ray density is proportional to irradiance can be found in Born and Wolf.