Geometrical optics
Design of imaging systems

“Geometrical optics is either very simple, or else it is very complicated”

Richard P. Feynman

What’s it good for?
1. Where is the image?
2. How large is it?
3. How bright is it?
4. What is the image quality?¹

“Useful when size of aperture is > 100 \( \lambda \)"

Why?

The ray equation

Approx. solution of ME when \( n(r) \) is slow

\[
\vec{E}(\vec{r}) = \mathbf{E}(\vec{r}) e^{-jk_0 S(\vec{r})}
\]

Assume slowly varying amplitude \( \mathbf{E} \) and phase \( S \)

\[
k_0 S(\vec{r}) = k_x x + k_y y + k_z z
\]

E.g. plane wave

\[
k_0 S(\vec{r}) = k_0 \sqrt{x^2 + y^2 + z^2}
\]

E.g. spherical wave

Substitute into isotropic wave equation and retain lowest terms…

\[
|\nabla S(\vec{r})|^2 = n^2(\vec{r})
\]

Ray equation – reduction of Maxwell’s equations.

Contours of \( S(r) \) at multiples of \( 2\pi \)

“Ray” = curve \( \perp \) to \( S(r) \)

\[
S(r) \quad \text{Optical path length} \quad \frac{B}{A} n(\vec{r}) ds \quad [\text{m}]
\]

Robert McLeod
The eikonal equation
An equation for evolution of ray trajectory

Take square-root of ray equation,
\[ \nabla S(\vec{r}) = n(\vec{r})\hat{s} = n(\vec{r})\frac{d\vec{r}}{ds} \]

then take derivative in \( s \),
\[ \frac{d}{ds}[\nabla S(\vec{r})] = \frac{d}{ds}\left[n(\vec{r})\frac{d\vec{r}}{ds}\right] \]

and chain rule for \( \nabla \),
\[ \frac{d}{ds}[\nabla S(\vec{r})] = \frac{d\vec{r}}{ds} \cdot \nabla[\nabla S(\vec{r})] = \left[\frac{\nabla S(\vec{r})}{n(\vec{r})}\right] \cdot \nabla[\nabla S(\vec{r})] \]

finally apply another \( \nabla \) identity.
\[ = \frac{1}{2n(\vec{r})} \nabla[\nabla S(\vec{r}) \cdot \nabla S(\vec{r})] = \frac{1}{2n(\vec{r})} \nabla[n^2(\vec{r})] = \nabla n(\vec{r}) \]

\[ \frac{d}{ds}\left[n(\vec{r})\frac{d\vec{r}}{ds}\right] = \nabla n(\vec{r}) \]

\( s \) Parametric distance along ray \([\text{m}]\)
\( \vec{r}(s) \) Ray trajectory \([\text{m}]\)
Ray trajectory $n=\text{constant}$

Finally!

\[
\frac{d}{ds} \left[ n(\vec{r}) \frac{d\vec{r}}{ds} \right] = \nabla n(\vec{r}) \quad \text{Eikonal}
\]

\[
\frac{d}{ds} \left[ n_0 \frac{d\vec{r}}{ds} \right] = \nabla n_0 = \vec{0} \quad n(r) = n_0, \text{ a constant}
\]

\[
\vec{r}(s) = \vec{c} \cdot s \quad \text{where } \vec{c} \text{ is a constant.}
\]

Thus, after ~40 slides, we have discovered that, in homogenous materials, rays travel in straight lines!
Pinhole camera

- Mo Ti, China, invented 5th century BC
- Aristotle observes image of eclipse cast through leaves, 300 BC
  - Aristotle formulates theory of light and color (some right, some wrong)
- Alhazen (Ibn Al-Haytham) invented CA 1000 AD
- Della Porta invented CA 1600
- Kepler named it *Camera Obscura* and suggested lens for efficiency, 1604

Magnification, $M = - l' / l$

- Very large depth of focus
- Very small chromatic effects
- Very poor power efficiency
- Dr. Webster Cash at CU funded by NASA Institute for Advanced Concepts to examine pinhole camera in space as extrasolar planet imager.

Abelardo Morell pinhole image of Manhattan
Fermat’s principle
Another (important) form of the eikonal

Optical path length \( S = \int_{A}^{B} n(\vec{r}) ds \)

- Propagation time = \( S / c \)
- Hero of Alexandria (in *Catoptrica*, ca 50 AD): “Light travels in straight lines”
- Pierre de Fermat (ca 1650): “Light travels the path which takes the minimum time.”
- Correct: “The time of travel is stationary:”

\[ \partial \int_{A}^{B} n(\vec{r}) ds = 0 \]

- Example 1: Concave mirror with radius of curvature < ellipse
- Example 2: Lifeguard problem
- See Born and Wolf for a derivation from Maxwell’s equations

- Why do we care? At an image point, all OPL must be equal.
Radiometry

Review of terminology & inverse square law

\[ \Omega = \frac{A}{R^2} \]
Solid angle \( \Omega \) is area of sphere subtended A over radius of sphere \( R^2 \)

\[ I = \frac{\phi}{\Omega} \]
Intensity I of point source is power \( \phi \) emitted into solid angle \( \Omega \)

\[ E = \frac{\phi}{A} \]
Irradiance E on surface is power \( \phi \) per unit area A

\[ = \frac{I}{R^2} \]
Irradiance of surface by a point source is given by intensity I of point source over distance to surface \( R^2 \)

“Inverse square law”

<table>
<thead>
<tr>
<th>Q</th>
<th>Energy</th>
<th>[J]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi )</td>
<td>Power (flux)</td>
<td>[J/s = W]</td>
</tr>
<tr>
<td>I</td>
<td>Intensity</td>
<td>[W/sr]</td>
</tr>
<tr>
<td>E</td>
<td>Irradiance</td>
<td>[W/m^2]</td>
</tr>
<tr>
<td>L</td>
<td>Radiance (photometric “brightness”)</td>
<td>[W/(sr m^2)]</td>
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Radiometry via rays

Consider a finite source radiating into a cone.

\[ E = \frac{\phi}{\delta A} = \frac{\phi}{4\pi R^2} \]

Irradiance E on surface \( \delta A \) given power \( \phi \) into cone.

Now launch a set of rays inside this cone and examine the ray density as a function of radius.

\[ \rho \equiv \frac{\delta N}{\delta A} = \frac{\delta N}{4\pi R^2} \propto E \]

A general proof that ray density is proportional to irradiance can be found in Born and Wolf.
Postulates of geometrical optics

- Rays are normal to equi-phase surfaces (wavefronts)
- The optical path length between any two wavefronts is equal
- The optical path length is stationary with respect to the variables that specify it
- Rays satisfy Snell’s laws of refraction and reflection
- The irradiance at any point is proportional to the ray density at that point
Sign convention
Bookkeeping for cascaded systems

• Light travels left to right
• Distances to the right are +
• Heights above the axis are +
• The radius of a surface is + if its center of curvature is to the right
• Focal lengths of converging elements are +
• Light traveling in a – direction (after a mirror) uses distances and indices that are negative
Sign convention

Graphically

- Design of ideal imaging systems with geometrical optics
  - Paraxial, thin lenses and graphical ray tracing