Maxwell’s equations

The fine print of “Let there be light”

\[
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{Faraday’s law}
\]

\[
\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \quad \text{Ampere’s law}
\]

\[
\nabla \cdot \vec{B} = 0 \quad \text{Gauss’ laws}
\]

\[
\nabla \cdot \vec{D} = \rho
\]

<table>
<thead>
<tr>
<th>\vec{E}</th>
<th>Electric field</th>
<th>[V/m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>\vec{H}</td>
<td>Magnetic field</td>
<td>[A/m]</td>
</tr>
<tr>
<td>\vec{D}</td>
<td>Electric flux density</td>
<td>[C/m^2]</td>
</tr>
<tr>
<td>\vec{B}</td>
<td>Magnetic flux density</td>
<td>[Wb/m^2]</td>
</tr>
<tr>
<td>\vec{J}</td>
<td>Electric current density</td>
<td>[A/m^2]</td>
</tr>
<tr>
<td>\rho</td>
<td>Electric charge density</td>
<td>[C/m^3]</td>
</tr>
<tr>
<td>\nabla \times</td>
<td>Curl</td>
<td>[1/m]</td>
</tr>
<tr>
<td>\nabla \cdot</td>
<td>Divergence</td>
<td>[1/m]</td>
</tr>
</tbody>
</table>
Constitutive relations
Interaction with matter

\[ \vec{D} = \varepsilon_0 \int_{-\infty}^{t} \varepsilon(t-\tau) \cdot \vec{\varepsilon}(\tau) \, d\tau \]

Dispersive & anisotropic

\[ \frac{\varepsilon \neq f(t)}{\varepsilon = f(t)} \rightarrow \varepsilon_0 \varepsilon \cdot \vec{\varepsilon}(\tau) \]

Anisotropic

\[ \varepsilon = \mu \rightarrow \varepsilon_0 \varepsilon \cdot \vec{\varepsilon}(\tau) \]

Isotropic

\[ \vec{B} = \mu_0 \int_{-\infty}^{t} \mu(t-\tau) \cdot \vec{H}(\tau) \, d\tau \rightarrow \mu_0 \vec{H} \]

Nonmagnetic

\[ \vec{J} = \sigma \cdot \vec{E} \]

Ohm's Law

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>\varepsilon_0</td>
<td>Permittivity of free space</td>
<td>8.854... \times 10^{-12}</td>
<td>[F/m]</td>
</tr>
<tr>
<td>\varepsilon</td>
<td>Dielectric constant</td>
<td></td>
<td></td>
</tr>
<tr>
<td>\mu_0</td>
<td>Permeability of free space</td>
<td>4 \pi 10^{-7}</td>
<td>[H/m]</td>
</tr>
<tr>
<td>\mu</td>
<td>Relative permeability</td>
<td></td>
<td></td>
</tr>
<tr>
<td>\sigma</td>
<td>Conductivity</td>
<td></td>
<td>[\Omega/m]</td>
</tr>
</tbody>
</table>
Boundary conditions
Fields at sharp change of material

These are derived from Maxwell’s equations.

In the absence of surface charge or current...

\[
\begin{align*}
\mathcal{E}_t^1 &= \mathcal{E}_t^2 & \text{Conservation of transverse electric and magnetic fields} \\
\mathcal{H}_t^1 &= \mathcal{H}_t^2 \\
\mathcal{D}_n^1 &= \mathcal{D}_n^2 & \text{Conservation of normal electric and magnetic flux densities} \\
\mathcal{B}_n^1 &= \mathcal{B}_n^2
\end{align*}
\]

Unit vector normal to boundary
\hat{n}

Unit vector transverse (or tangential) to boundary
\hat{t}
Monochromatic fields

Important simplification

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) e^{j\omega t} \, d\omega \]

Fourier Transform

\[ \mathcal{E} = \text{Re}(E e^{j\omega t}) \]

Monochromatic fields \( E \) transform like time-domain fields \( \mathcal{E} \) for linear operators

\[ \frac{d}{dt} \rightarrow j\omega \]

Removes all time-derivates.

\[ \nabla \times \vec{E} = -j\omega \vec{B} \]

\[ \nabla \times \vec{H} = +j\omega \vec{D} + \vec{J} \]

Monochromatic Maxwell’s equations.

\[ \nabla \cdot \vec{B} = 0 \]

\[ \nabla \cdot \vec{D} = \rho \]
Monochromatic constitutive relations

The reason for using the monochromatic assumption

### Convolution

\[
\vec{D} = \mathcal{E}_0 \int_{-\infty}^{t} \mathcal{E}(t-\tau) \cdot \vec{\mathcal{E}}(\tau) \, d\tau \quad \Rightarrow \quad \vec{D} = \mathcal{E}_0 \mathcal{E}(\omega) \cdot \vec{E}
\]

### Multiplication

\[
\vec{B} = \mu_0 \int_{-\infty}^{t} \mu(t-\tau) \cdot \vec{\mathcal{H}}(\tau) \, d\tau \quad \Rightarrow \quad \vec{B} = \mu_0 \mu(\omega) \cdot \vec{H}
\]

\[
\mathcal{E}(\omega) = \int_{0}^{\infty} \mathcal{E}(t) e^{-j\omega t} \, dt
\]

### Inverse Fourier Transform

Note that \(\mathcal{E}\) is now \(f(\omega)\) & not \(f(t)\).

If \(\mathcal{E}\) is not constant in \(\omega\), it causes “dispersion” of pulses.

### Conditions for lossless materials derived from Poynting vector (next)

\[
\mathcal{E}^+ = \mathcal{E}
\]

\[
\mu^+ = \mu
\]

\(\mathcal{E}^+\) is the Hermitian conjugate:

\[
\mathcal{E}_{ji} \rightarrow \mathcal{E}_{ij}^*
\]
Complex dielectric tensor
For conductive materials

\[ \nabla \times \vec{H} = + j \omega \vec{D} + \vec{J} \]

Ampere’s law

\[ = j \omega \varepsilon_0 \varepsilon \cdot \vec{E} + \sigma \cdot \vec{E} \]

Constitutive relations

\[ = j \omega \varepsilon_0 \left( \varepsilon - \frac{j}{\omega \varepsilon_0} \sigma \right) \cdot \vec{E} \]

Group terms

Complex dielectric tensor

From this point on the dielectric tensor will be taken to be complex via this definition.
### Poynting vector

**Power flow**

\[ \vec{P} = \vec{E} \times \vec{H} \]

- **Instantaneous power flow**

\[ \langle \vec{P} \rangle = \frac{1}{T} \int_0^T \vec{P} \, dt \]

- **Time-averaged power flow**

\[ = \frac{1}{2} \text{Re}(\vec{E} \times \vec{H}) \]

\[ = \text{Re}(\vec{P}) \]

- **Define complex vector**

\[ \vec{P} = \frac{1}{2} \vec{E} \times \vec{H} \]

- **Real part of P is \( \langle \vec{P} \rangle \)**

| \( \mathcal{P} \) | Power per unit area | [W/m²] |
Wave equation

Eliminate all fields but $E$

\[
\nabla \times \nabla \times \vec{E} = -j \omega \nabla \times \vec{B} \quad \text{Take curl of Faraday's law}
\]
\[
= -j \omega \mu_0 \nabla \times \vec{H} \quad \text{Magnetic constitutive}
\]
\[
= \omega^2 \mu_0 \vec{D} \quad \text{Ampere’s law}
\]
\[
= \omega^2 \varepsilon_0 \mu_0 \varepsilon \cdot \vec{E} \quad \text{Electric constitutive}
\]

\[
\nabla \times \nabla \times \vec{E} - k_0^2 \varepsilon \cdot \vec{E} = 0 \quad \text{Monochromatic WE}
\]

\[
\nabla^2 \vec{E} + k^2 \vec{E} = 0 \quad \text{Scalar simplification}
\]

<table>
<thead>
<tr>
<th>$k_0$</th>
<th>Wave number of free space</th>
<th>$\omega/c = 2\pi/\lambda_0$ [1/m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>Speed of light in vacuum</td>
<td>$1/\sqrt{\mu_0\varepsilon_0}$ [m/s]</td>
</tr>
</tbody>
</table>
Plane-wave solution

Cartesian eigensolution in $\infty$ homogeneous space.

Assume a plane-wave solution. Note that this is an eigenfunction.

$$\vec{\varepsilon} = \text{Re} \left( \vec{E}_0 \ e^{j(\alpha x - \vec{k} \cdot \vec{r})} \right)$$

or

$$\vec{E} = \vec{E}_0 \ e^{-j \vec{k} \cdot \vec{r}}$$

Similar to monochromatic assumption, removes all space derivatives.

This transforms the wave equation into three coupled linear equations (one for each vector component of $E$) in three variables (the three components of $k$). To have a non-trivial solution, the determinant of this matrix equation must be zero.

$$\left| \left( \vec{k} \times \vec{I} \right)^2 + k_0^2 \vec{\varepsilon} \right| = 0$$

Characteristic equation.

By using Gauss’ Law, we can reduce this from 6th order in $n = k/k_0$ to 4th in $n$ (actually second order in $n^2$).

$$\vec{k} \cdot \vec{\varepsilon} \cdot \vec{k} \left( \frac{k}{k_0} \right)^4 + \vec{k} \cdot \left[ \text{adj} \vec{\varepsilon} - \left( \text{Tr} \text{adj} \vec{\varepsilon} \right) \vec{I} \right] \cdot \vec{k} \left( \frac{k}{k_0} \right)^2 + |\vec{\varepsilon}| = 0$$

<table>
<thead>
<tr>
<th>$n$</th>
<th>Index of refraction</th>
<th>$k/k_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>adj A</td>
<td>Adjoint of matrix</td>
<td>$</td>
</tr>
<tr>
<td>Tr A</td>
<td>Trace of matrix</td>
<td>$\Sigma A_{ii}$</td>
</tr>
</tbody>
</table>
Special cases

Isotropic and uniaxial

\[ \mathbf{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & & \\ & \varepsilon_{11} & \\ & & \varepsilon_{11} \end{bmatrix} \]

Isotropic material

\[ \left( \frac{k}{k_0} \right)^4 - 2\varepsilon_{11} \left( \frac{k}{k_0} \right)^2 + \varepsilon_{11}^2 = 0 \]

Characteristic equation

\[ n = \left\{ \pm \sqrt{\varepsilon_{11}}, \pm \sqrt{\varepsilon_{11}} \right\} \]

Four solutions, 2+, 2-

\[ \mathbf{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & & \\ & \varepsilon_{11} & \\ & & \varepsilon_{33} \end{bmatrix} \]

Uniaxial material in principal coords.

\[ \hat{k} = \cos \theta \sin \varphi \hat{x} + \sin \theta \sin \varphi \hat{y} + \cos \varphi \hat{z} \]

Propagation direction

\[ \left( \frac{k}{k_0} \right)^4 - \varepsilon_{11} \left( \frac{k}{k_0} \right)^2 + \frac{\varepsilon_{11}^2 \varepsilon_{33}}{\varepsilon_{11} \sin^2 \varphi + \varepsilon_{33} \cos^2 \varphi} = 0 \]

Characteristic equation

\[ n = \left\{ \pm \sqrt{\varepsilon_{11}}, \pm \left( \frac{\cos^2 \varphi}{\varepsilon_{11}} + \frac{\sin^2 \varphi}{\varepsilon_{33}} \right)^{-1/2} \right\} \]

Four solutions,
2 ordinary
2 extraordinary

“Index ellipsoid”
• Physical nature of light
  – Plane waves

Uniaxial surface w/ pols
Biaxial surface w/ polys

- Physical nature of light
  - Plane waves
Cartesian eigensolution at $\infty$ half-space

Foundation of Fourier optics

Break transmitted wave vector into normal and transverse components.

$$\vec{k}_{\text{transmit}} = k_n \hat{n} + k_t \hat{t}$$

Excitation on boundary.

$$\vec{\tilde{E}}_t = E_t \hat{e} e^{j(\omega \vec{r} - \vec{k}_t \cdot \vec{r})}$$

Transverse wave vector conserved.

$$k_t = \vec{k}_{\text{inc}} \cdot \hat{t} = k_{\text{inc}} \sin \theta_{\text{inc}} = \vec{k}_{\text{trans}} \cdot \hat{t} = k_{\text{trans}} \sin \theta_{\text{trans}}$$

Problem: Given orientation of boundary ($\hat{n}, \hat{t}$), material ($\varepsilon$), and boundary excitation ($\vec{\tilde{E}}_t$), how does plane wave propagate into material ($k_n$)?
Booker quartic

Characteristic equation for Fourier optics

\[ \vec{k}_{\text{transmit}} = k_n \hat{n} + k_t \hat{t} \]

Plug into characteristic equation.

…and solve for \( k_n \), yielding new characteristic equation.

Note that this is now a general 4th order equation

\[ a_4 \left( \frac{k_n}{k_0} \right)^4 + a_3 \left( \frac{k_n}{k_0} \right)^3 + a_2 \left( \frac{k_n}{k_0} \right)^2 + a_1 \left( \frac{k_n}{k_0} \right) + a_0 = 0 \]

The coefficients are given in terms of the known variables:

\[ a_4 = \hat{n} \cdot \vec{\varepsilon} \cdot \hat{n} \]

\[ a_3 = \hat{n}_t \cdot \left( \begin{array}{c} \varepsilon \\ \varepsilon = T \\ \varepsilon + \varepsilon \end{array} \right) \cdot \hat{n} \]

\[ a_2 = \hat{n} \cdot \left[ \text{adj } \varepsilon - \left( \text{Tr adj } \varepsilon \right) I \right] \cdot \hat{n} + \hat{n}_t \cdot \varepsilon \cdot \hat{n}_t + n_t^2 a_4 \]

\[ a_1 = \hat{n}_t \cdot \left[ \text{adj } \varepsilon + \text{adj } \left( \begin{array}{c} \varepsilon \\ \varepsilon = T \end{array} \right) \right] \cdot \hat{n} + n_t^2 a_3 \]

\[ a_0 = \hat{n}_t \cdot \left[ \text{adj } \varepsilon - \left( \text{Tr adj } \varepsilon \right) I \right] \cdot \hat{n}_t + n_t^2 \hat{n}_t \cdot \varepsilon \cdot \hat{n}_t + || \varepsilon || \]
Isotropic refraction

a.k.a. Snell’s law – gives ray directions at boundary

\[ \hat{n} \]

Real space

\[ \frac{\varepsilon}{\varepsilon_0} = \frac{\hat{k}_n}{\hat{k}_t} \]

\[ \theta_{inc} \]

\[ \theta_{trans} \]

\[ \theta_{refl} \]

\[ n_{trans} = \sqrt{\varepsilon - n_t^2} \]

\[ = n_{inc} \sin \theta_{inc} \]

\[ = n_{trans} \sin \theta_{trans} \]

\[ = n_{refl} \sin \theta_{refl} \]

Fourier space

Real part

Imag. part
Total internal reflection

a.k.a. evanescent waves

Real space

\[ \beta = \frac{\epsilon}{\epsilon_0} \]

\[ n\hat{t} \]

\[ \theta_{inc} \]

\[ \theta_{refl} \]

\[ k_{inc} \]

\[ k_{refl} \]

\[ \alpha_n \]

\[ E_{trans} = E_0 e^{k_0(-j n\hat{t} - \alpha_n \hat{n}) \cdot \vec{r}} \]

Momentum space

\[ \alpha_n \]

\[ n_{inc} \]

\[ n_{refl} \]

\[ \theta_{inc} \]

\[ \theta_{refl} \]

\[ n_t \]

\[ n_{inc} \sin \theta_{inc} = n_{refl} \sin \theta_{refl} \]
Extraordinary refraction
and other fun with crystals

Real space

Momentum space

Physical nature of light
– Plane waves

Saleh & Teich 6.3
**Materials with loss**
e.g. polarizers

\[
\varepsilon = \begin{bmatrix}
1.5^2 & 1.5^2 \\
1.5^2 & 1.8^2 - j2
\end{bmatrix}
\]

Uniaxial crystal with loss only in \( z \). Lossless part same as previous example.

- Physical nature of light
  - Plane waves

- Ordinary ray polarization \( \perp \) to \( z \), so not changed.
- Extraordinary ray polarization \( \perp \) to \( z \) at \( c \), so unchanged there.
- \( z \) component of extraordinary ray increases away from \( c \), as does loss.
Fresnel Coefficients

Amplitude and phase of waves at boundary

\[
\begin{align*}
\text{s ("senkrecht") / TE / \perp} & \\
\frac{r_\perp}{E_i} & = \frac{n_1 \cos \theta - n_2 \cos \theta'}{n_1 \cos \theta + n_2 \cos \theta'} = -\frac{\sin(\theta - \theta')}{\sin(\theta + \theta')} \\
\frac{t_\perp}{E_i} & = \frac{2n_1 \cos \theta}{n_1 \cos \theta + n_2 \cos \theta'} = \frac{2\sin \theta' \cos \theta}{\sin(\theta + \theta')} \\
\text{p ("parallel") / TM / \parallel} & \\
\frac{r_\parallel}{E_i} & = \frac{n_2 \cos \theta - n_1 \cos \theta'}{n_2 \cos \theta + n_1 \cos \theta'} = \frac{\tan(\theta - \theta')}{\tan(\theta + \theta')} \\
\frac{t_\parallel}{E_i} & = \frac{2n_1 \cos \theta}{n_2 \cos \theta + n_1 \cos \theta'} = \frac{2\sin \theta' \cos \theta}{\sin(\theta + \theta') \cos(\theta - \theta')} 
\end{align*}
\]
Fresnel Coefficients

Special values

\[ R = |r|^2 = \left( \frac{n_1 - n_2}{n_1 + n_2} \right)^2 = \left( \frac{\Delta n}{2n} \right)^2 \]

Normal incidence

\[ \theta_B = \tan^{-1}\left( \frac{n_2}{n_1} \right) \]

Brewster’s angle

\[ \theta + \theta' = \frac{\pi}{2} \]

Physical interpretation of \( \theta_B \):
Dipoles excited in \( n_2 \) can not radiate in direction of reflected wave when it is \( \perp \)
Fresnel Coefficients

Phase and TIR

\[ n_1 > n_2 \]

TIR

\[ \theta_C = \sin^{-1}\left(\frac{n_2}{n_1}\right) \]

\[ \Delta \phi = \begin{cases} \pi & \text{for } n_2 > n_1 \\ 0 & \text{for } n_2 < n_1 \end{cases} \]

Phase of TE electric field on reflection.

TM has the opposite (same) sign < (>) \( \theta_B \).

“Goos-Hanchen” phase-shift
Propagation and diffraction
Regions and their naming

\[ z = \frac{L^2}{\lambda} \]

- Near field
- Fresnel
- Fraunhofer or far field

\( L = 20 \, \mu m \)
\( \lambda = 1 \, \mu m \)
Fourier optics in 1 equation
Valid in all regions

\[ E(t, x, y, z) = \mathcal{F}_{t,xy}^{-1} \left\{ \mathcal{F}_{t,xy} \left[ E(t, x, y, 0) e^{-jk_z(\omega, k_T)z} \right] \right\} \]

Where \( k_z(\omega, k_T) \) is given by the Booker quartic.

The Fourier transform (in case you’ve forgotten) is

\[
\mathcal{F}_{t,xy} \equiv \int dt \int dx \, dy \ e^{-j(\omega t - (k_x x + k_y y))}
\]

\[
\mathcal{F}_{t,xy}^{-1} \equiv \int d\omega \int dk_x \, dk_y \ e^{j(\omega t - (k_x x + k_y y))}
\]
First imaging limitation
Band-limiting propagation in free space

Spatial frequencies beyond TIR do not propagate. Thus only a few wavelengths after the object, the radiated field will be band-limited. For example, a 1D rect function:

\[
\mathcal{F}\left\{ \text{rect}\left( \frac{x}{L} \right) \right\} = 2 \text{sinc}\left( \frac{1}{2} k_x L \right) = 2 \text{sinc}(\pi f_x L) = 2 \text{sinc}\left( \pi \frac{L}{\lambda_x} \right)
\]

The highest spatial frequency that can be transmitted is thus

\[ f_x = \frac{1}{\lambda} \text{ or } k_x = \frac{2\pi}{\lambda} \]
Near-field region
~no diffraction

How far will the light from a rectangular aperture propagate before it begins to diffract? Calculate the phase accumulated between DC and first null and assume when this reaches $\pi$ then the beam will look significantly different:

$$\Delta k_z z = k_z (k_x = 0) z - k_z (k_x = \frac{2\pi}{L}) z$$

Phase accumulated vs. $z$

$$= k z - z \sqrt{k^2 - \left(\frac{2\pi}{L}\right)^2}$$

Isotropic $k_z$

$$\approx k z - z \left(k + \frac{1}{2k} \left(\frac{2\pi}{L}\right)^2\right)$$

Binomial expansion

$$= \lambda \frac{\pi}{L^2} z$$

At null spatial frequencies now $\pi$ out of phase

$$\equiv \pi$$

Rayleigh range aka near-field/far-field boundary

$$\therefore z = \frac{L^2}{\lambda}$$

Thus a band-limited image can be transmitted directly through free space for a distance $< L^2 / \lambda$
Beyond the near-field

Optical Field

\[ F\{\text{Rect}(x/L)\} = 2 \text{ Sinc}(k_x L/2) \]

\[ k_x = k_0 \sin(\theta) \]

First null is at

\[ \sin(\theta) = \lambda/L = 1/50 = 0.125/5.8 \]

E at end of propagation

- Physical nature of light
  - Propagation and diffraction
Diffraction from a 1D lens

Optical Field

\[ \mathfrak{F}\{\text{Rect}(x/L)\} = L \text{ Sinc}(k_xL/2) \]

\[ k_x = k_0 \sin(\theta) \]

First null is at \( x = f \lambda/L = 5800 \times 1/200 = 29 \)
Airy disk

The diffraction limited resolution

Q: What is the electric field at the focus of a uniform amplitude cone?

Real space

\[ E(r) = 2 J_1 \left( \frac{2 \pi}{\lambda} r \sin \theta_0 \right) \left/ \frac{2 \pi}{\lambda} r \sin \theta_0 \right. \]

Fourier space

\[ E(k_r) = \text{circ} \left[ k_r \left/ \left( \frac{2 \pi}{\lambda} \sin \theta_0 \right) \right. \right] \]

Diameter of first null:

\[ r \left/ \frac{2 \pi}{\lambda} \sin \theta_0 = 3.83171... \right. \]

\[ D = 2r \approx 1.22 \frac{\lambda}{\sin \theta_0} \]

\[ \theta_0 \]

\[ \theta_0 \]

\[ E(0) = 2 J_1 \left( \frac{2 \pi}{\lambda} \right) \left/ \frac{2 \pi}{\lambda} \right. \]

Peak E Energy in ring

<table>
<thead>
<tr>
<th>( J_1(r) = 0 )</th>
<th>Peak E</th>
<th>Energy in ring</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.83171...</td>
<td>+1.0</td>
<td>83.9%</td>
</tr>
<tr>
<td>7.01559...</td>
<td>-0.017</td>
<td>7.1%</td>
</tr>
<tr>
<td>10.1735...</td>
<td>+0.0041</td>
<td>2.8%</td>
</tr>
<tr>
<td>13.3237...</td>
<td>-0.0016</td>
<td>1.5%</td>
</tr>
</tbody>
</table>
Fresnel integral
Convolution with impulse response

The transfer function of free space is

\[ H(\omega, k_x, k_y; z) \equiv \frac{E(\omega, k_x, k_y, z)}{E(\omega, k_x, k_y, 0)} = e^{-jz\sqrt{\left(\frac{\omega}{c}\right)^2 - k_x^2 - k_y^2}} \]

The paraxial approximation to this is found by expanding the square root in a binomial series. Note that this is the solution to the paraxial wave equation (Helmholtz equation):

\[ H(\omega, k_x, k_y; z) \approx e^{\frac{jk_0 k_x^2 + k_y^2}{2k_0} - jk_0 z} \]

The transverse spatial inverse Fourier transform of this is the paraxial monochromatic impulse response of free space

\[ h(\omega, x, y; z) = \mathcal{F}^{-1}_{k_x, k_y} \{ H(\omega, k_x, k_y; z) \} = \frac{jk_0}{2\pi z} e^{-jk_0 \frac{x^2 + y^2}{2z}} \]

Yielding the Fresnel diffraction formula

\[ E(\omega, x, y, z) = \frac{jk_0}{2\pi z} e^{-jk_0 z} \int \int E(\omega, \xi, \psi, 0) e^{-jk_0 \frac{(x-\xi)^2 + (y-\psi)^2}{2z} - jk_0 z} \, d\xi \, d\psi \]