2. Vector waves and polarization

2.1 Electromagnetic plane wave

\[ \mathbf{E} = \mathbf{E}(\mathbf{r} \cdot \mathbf{\hat{z}} - vt) \quad \mathbf{H} = \mathbf{H}(\mathbf{r} \cdot \mathbf{\hat{z}} - vt) \]

where \( \mathbf{\hat{z}} \) is a unit vector in the direction of propagation.

From Maxwell and the material equations we get

\[ \mathbf{E} = -\sqrt{\frac{\mu}{\varepsilon}} \mathbf{\hat{z}} \times \mathbf{H} \]
\[ \mathbf{H} = -\sqrt{\frac{\varepsilon}{\mu}} \mathbf{\hat{z}} \times \mathbf{E} \]

so \( \mathbf{E}, \mathbf{H},\mathbf{k} \) are a right-handed orthogonal triad. Moreover, \( \sqrt{\varepsilon|\mathbf{E}|} = \sqrt{\mu|\mathbf{H}|} \). Thus, the Poynting vector is

\[ \mathbf{S} = \frac{c}{4\pi} \sqrt{\frac{\varepsilon}{\mu}} |\mathbf{E}|^2 \mathbf{\hat{z}} = \frac{c}{4\pi} \sqrt{\frac{\mu}{\varepsilon}} |\mathbf{H}|^2 \mathbf{\hat{z}} = \nu \mathbf{\hat{w}} \mathbf{\hat{z}} \]

where \( \nu \) is the speed of light in the medium and \( w \) is the total energy density. Therefore, the Poynting vector is interpreted as the flow of energy.

2.2 Harmonic plane waves

The harmonic electromagnetic plane wave can be written as

\[ \mathbf{E}(z, t) = \text{Re}\left\{ \mathbf{A} \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)] \right\} \]

where \( \mathbf{A} = A_x \mathbf{\hat{x}} + A_y \mathbf{\hat{y}} \) and \( A_{x,y} = a_{x,y} \exp(i\varphi_{x,y}) \).

Let us now observe the behavior in space of the tip of the electric vector. Each cartesian component satisfies

\[ E_{x,y} = a_{x,y} \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \varphi_{x,y}) \]

Thus, the tip of the electric field describes the following ellipse

\[ \left( \frac{E_x}{a_x} \right)^2 + \left( \frac{E_y}{a_y} \right)^2 - 2 \frac{E_x E_y}{a_x a_y} \cos \varphi = \sin^2 \varphi, \quad \varphi = \varphi_y - \varphi_x \]

\( H_x, H_y \) satisfy an analogous set of equations.
In a set of coordinates along the axes of the ellipse \((x', y')\), the parameters of the ellipse can be determined from the following equations:

\[
\begin{align*}
a_{x'}^2 + a_{y'}^2 &= a_x^2 + a_y^2 \\
a_{y'}/a_{x'} &= \pm \tan \chi \quad (-\frac{\pi}{4} < \chi \leq \frac{\pi}{4}) \\
\sin 2\chi &= \frac{2a_x a_{y'}}{a_x^2 + a_y^2} \sin \varphi \\
\tan 2\psi &= \frac{2a_x a_{y'}}{a_x^2 - a_y^2} \cos \varphi \quad (0 \leq \psi < \pi)
\end{align*}
\]

\(a_x, a_y\) are half of the minor and major axis, \(\psi\) is the angle of the major axis with the Ox axis, and \(\chi\) is an angle that determines the ellipticity. The state of polarization is determined by \(\psi\) and \(\chi\).

We say the light is right-handed (RH) \((\sin \varphi > 0)\) or left-handed (LH) polarized according to the direction of the trajectory of the electric vector as seen from the direction of the propagation:

For \(z=\text{const.}\), the tip of the vector describes an ellipse
For \(t=\text{const.}\) the tip describes a helix

**Linear Polarization**

\[\varphi = m\pi, \; m \text{ integer}\]

**Circular polarization**

\[a_x = a_y = a\]

\[\varphi = (2m + 1)\pi / 2, \; m \text{ integer}\]

\[E_x^2 + E_y^2 = a^2\]
Stokes parameters

We would like to describe the polarization ellipse by three parameters with the same dimensions. Stokes proposed to use quadratic parameters in the field strength. These parameters can be easily determined by intensity measurements with polarizers and quarter wave plates.

\[ s_0 = a_x^2 + a_y^2 \]  
\[ s_1 = a_x^2 - a_y^2 \]  
\[ s_2 = 2a_x a_y \cos \varphi \]  
\[ s_3 = 2a_x a_y \sin \varphi \]

Only three of the four parameters are independent since \( s_0^2 = s_1^2 + s_2^2 + s_3^2 \).

In terms of \( \psi \) and \( \chi \):

\[ s_1 = s_0 \cos 2\chi \cos 2\psi \]
\[ s_2 = s_0 \cos 2\chi \sin 2\psi \]
\[ s_3 = s_0 \sin 2\chi \]

\( s_1, s_2, s_3 \) are the cartesian coordinates of a point on a sphere of radius \( s_0 \). \( 2\chi, 2\psi \) are the spherical coordinates. There is a one-to-one correspondence between all the possible states of polarization of a plane monochromatic wave and the points of the sphere (Poincaré sphere).

The polarization is linear along the equator. The orientation is horizontal at the intersection with the axis \( OS_1 (\psi = 0) \), at \( 45^\circ \) at the intersection with \( OS_2 (\psi = \pi / 2) \), and at \( 90^\circ \) in the intersection with the negative part of the \( OS_1 \) axis (\( \psi = \pi \)). In the upper hemisphere we have right-handed elliptical polarization, and in particular right-handed circular (RHC) polarization at the pole. In the bottom hemisphere we have left-handed elliptical polarization and left-handed circular polarization (LHC) at the
pole. Along one meridian we evolve, from LHC polarization in the bottom pole to RHC polarization in the upper pole, through all the degrees of ellipticity but having a constant orientation of the major axis.