8.2.1 Example: Transfer Functions of the Buck-Boost Converter

The small-signal ac equations of the buck-boost converter are derived in Section 7.2, with the result [Eq. 7.42] repeated below:

\[
L \frac{d \hat{i}(t)}{dt} = D \hat{v}_g(t) + D' \hat{v}(t) + \left(V_g - V\right) \hat{d}(t)
\]
\[
C \frac{d \hat{v}(t)}{dt} = -D' \hat{i}(t) - \frac{\hat{v}(t)}{R} + I \hat{d}(t)
\]
\[
\hat{i}_g(t) = D \hat{i}(t) + I \hat{d}(t)
\]  

(8.106)

The converter contains two independent ac inputs: the control input \( \hat{d}(s) \) and the line input \( \hat{v}_g(s) \). The ac output voltage variations \( \hat{v}(s) \) can be expressed as the superposition of terms arising from these two inputs:

\[ \hat{v}(s) = G_{vd}(s) \hat{d}(s) + G_{vg}(s) \hat{v}_g(s) \]

(8.107)

Hence, the transfer functions \( G_{vd}(s) \) and \( G_{vg}(s) \) can be defined as

\[ G_{vd}(s) = \left. \frac{\hat{v}(s)}{\hat{d}(s)} \right|_{\hat{v}_g(s) = 0} \quad \text{and} \quad G_{vg}(s) = \left. \frac{\hat{v}(s)}{\hat{v}_g(s)} \right|_{\hat{d}(s) = 0} \]

(8.108)

An algebraic approach to deriving these transfer functions begins by taking the Laplace transform of Eq. (8.106), letting the initial conditions be zero:

\[
sL \hat{i}(s) = D \hat{v}_g(s) + D' \hat{v}(s) + \left(V_g - V\right) \hat{d}(s)
\]
\[
sC \hat{v}(s) = -D' \hat{i}(s) - \frac{\hat{v}(s)}{R} + I \hat{d}(s)
\]  

(8.109)

To solve for the output voltage \( \hat{v}(s) \), we can use the top equation to eliminate \( \hat{i}(s) \):

\[ \hat{i}(s) = \frac{D \hat{v}_g(s) + D' \hat{v}(s) + \left(V_g - V\right) \hat{d}(s)}{sL} \]

(8.110)

Substitution of this expression into the lower equation of (8.109) leads to

\[
sC \hat{v}(s) = \frac{D'}{sL} \left( D \hat{v}_g(s) + D' \hat{v}(s) + \left(V_g - V\right) \hat{d}(s) \right) - \frac{\hat{v}(s)}{R} + I \hat{d}(s)
\]

(8.111)

Solution for \( \hat{v}(s) \) results in

\[ \hat{v}(s) = \frac{-DD'}{D^2 + s \frac{L}{R} + s^2 LC} \hat{v}_g(s) - \frac{D\left(V_g - V\right) - sLI}{D^2 + s \frac{L}{R} + s^2 LC} \hat{d}(s) \]

(8.112)
We aren’t done yet —the next step is to manipulate these expressions into normalized form, such that the coefficients of $s^0$ in the numerator and denominator polynomials are equal to one:

$$\hat{v}(s) = \left( -\frac{D}{D'} \right) \frac{1}{1 + sL + s^2LC} \hat{v}_g(s) = \left( \frac{V_g - V}{D'} \right) \left( \frac{1 - s\frac{LI}{D'[V_g - V]}}{1 + sL + s^2LC} \right) \hat{d}(s)$$

This result is similar in form to Eq. (8.107). The line-to-output transfer function is

$$G_{vg}(s) = \frac{\hat{v}(s)}{\hat{v}_g(s) \mid \hat{d}(s) = 0} = \left( -\frac{D}{D'} \right) \frac{1}{1 + sL + s^2LC}$$

(8.113)

Thus, the line-to-output transfer function contains a dc gain $G_{g0}$ and a quadratic pole pair:

$$G_{vg}(s) = G_{g0} \frac{1}{1 + \frac{s}{Q\omega_0} + \left( \frac{s}{\omega_0} \right)^2}$$

(8.114)

Analytical expressions for the salient features of the line-to-output transfer function are found by equating like terms in Eqs. (8.114) and (8.115). The dc gain is

$$G_{g0} = -\frac{D}{D'}$$

(8.116)

By equating the coefficients of $s^2$ in the denominators of Eqs. (8.114) and (8.115), one obtains

$$\frac{1}{\omega_0^2} = \frac{LC}{D^2}$$

(8.117)

Hence, the angular corner frequency is

$$\omega_0 = \frac{D'}{\sqrt{LC}}$$

(8.118)

By equating coefficients of $s$ in the denominators of Eqs. (8.114) and (8.115), one obtains

$$\frac{1}{Q\omega_0} = \frac{L}{D'^2 R}$$

(8.119)

Elimination of $\omega_0$ using Eq. (8.118) and solution for $Q$ leads to

$$Q = D'R \sqrt{\frac{C}{L}}$$

(8.120)

Equations (8.116), (8.118), and (8.120) are the desired results in the analysis of the line-to-output transfer function. These expressions are useful not only in analysis situations, where it is desired to find numerical values of the salient features $G_{g0}$, $\omega_0$, and $Q$, but also in design situations, where it is desired to select numerical values for $R$, $L$, and $C$ such that given values of the salient features are obtained.
According to Eq. (8.113), the control-to-output transfer function is

$$G_{vd}(s) = \left| \frac{v(s)}{d(s)} \right|_{s=0} = \left( -\frac{V_g - V}{D'} \right) \frac{1 - s \frac{LI}{D'(V_g - V)}}{1 + s \frac{L}{D'^{2}R} + s^{2} \frac{LC}{D'^{2}}}$$

(8.121)

This equation is of the form

$$G_{vd}(s) = G_{d0} \left( 1 - \frac{s}{\omega_z} \right) \left( 1 + \frac{s}{Q \omega_0} + \left( \frac{s}{\omega_0} \right)^2 \right)$$

(8.122)

The denominators of Eq. (8.121) and (8.114) are identical, and hence $G_{vd}(s)$ and $G_{vg}(s)$ share the same $\omega_0$ and $Q$, given by Eqs. (8.118) and (8.120). The dc gain is

$$G_{d0} = -\frac{V_g - V}{D'} = -\frac{V_g}{D'^{2}} = \frac{V}{DD'}$$

(8.123)

The angular frequency of the zero is found by equating coefficients of $s$ in the numerators of Eqs. (8.121) and (8.122). One obtains

$$\omega_z = \frac{D'(V_g - V)}{LI} = \frac{D'^{2}R}{DL} \quad \text{RHP}$$

(8.124)

This zero lies in the right half-plane. Equations (8.123) and (8.124) have been simplified by use of the dc relations

$$V = -\frac{D}{D'} V_g$$

$$I = -\frac{V}{D'^{2}R}$$

(8.125)

Equations (8.118), (8.120), (8.123), and (8.124) constitute the results of the analysis of the control-to-output transfer function: analytical expressions for the salient features $\omega_0$, $Q$, $G_{d0}$, and $\omega_z$. These expressions can be used to choose the element values such that given desired values of the salient features are obtained.

Having found analytical expressions for the salient features of the transfer functions, we can now plug in numerical values and construct the Bode plot. Suppose that we are given the following values:

$$D = 0.6$$

$$R = 10 \, \Omega$$

$$V_g = 30 \, V$$

$$L = 160 \, \mu H$$
We can evaluate Eqs. (8.116), (8.118), (8.120), (8.123), and (8.124), to determine numerical values of the salient features of the transfer functions. The results are:

\[
\begin{align*}
|G_{g0}| &= \frac{D}{D'} = 1.5 \Rightarrow 3.5 \text{ dB} \\
|G_{d0}| &= \left| \frac{V}{DD'} \right| = 187.5 \text{ V} \Rightarrow 45.5 \text{ dBV} \\
f_0 &= \frac{\omega_0}{2\pi} = \frac{D'}{2\pi\sqrt{LC}} = 400 \text{ Hz} \\
Q &= DR\sqrt{\frac{C}{L}} = 4 \Rightarrow 12 \text{ dB} \\
f_z &= \frac{\omega_z}{2\pi} = \frac{D_{v0}^2R}{2\pi DL} = 2.65 \text{ kHz}
\end{align*}
\]

The Bode plot of the magnitude and phase of \(G_{vd}\) is constructed in Fig. 8.26. The transfer function contains a dc gain of 45.5 dBV, resonant poles at 400Hz having a \(Q\) of 4 \(\Rightarrow\) 12dB, and a right half-plane zero at 2.65 kHz. The resonant poles contribute \(-180^\circ\) to the high frequency phase asymptote, while the right half-plane zero contributes \(-90^\circ\). In addition, the inverting characteristic of the buck-boost converter leads to a \(180^\circ\) phase reversal, not included in Fig. 8.26.

The Bode plot of the magnitude and phase of the line-to-output transfer function \(G_{vg}\) is constructed in Fig. 8.27. This transfer function contains the same resonant poles at 400 Hz, but is missing the right half-plane zero. The dc gain \(G_{g0}\) is equal to the conversion ratio \(M(D)\) of the converter. Again, the \(180^\circ\) phase reversal, caused by the inverting characteristic of the buck-boost converter, is not included in Fig. 8.27.
### Table 8.2  
Salient features of the small-signal CCM transfer functions of some basic dc-dc converters

<table>
<thead>
<tr>
<th>Converter</th>
<th>$G_{V0}$</th>
<th>$G_{V0}$</th>
<th>$\omega_0$</th>
<th>$Q$</th>
<th>$\omega_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buck</td>
<td>$D$</td>
<td>$\frac{V}{D}$</td>
<td>$\frac{1}{\sqrt{LC}}$</td>
<td>$R \sqrt{\frac{C}{L}}$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>Boost</td>
<td>$\frac{1}{D}$</td>
<td>$\frac{V}{D}$</td>
<td>$\frac{D}{\sqrt{LC}}$</td>
<td>$D'R \sqrt{\frac{C}{L}}$</td>
<td>$\frac{D^2R}{L}$</td>
</tr>
<tr>
<td>Buck-boost</td>
<td>$-\frac{D}{D'}$</td>
<td>$\frac{V}{DD'}$</td>
<td>$\frac{D}{\sqrt{LC}}$</td>
<td>$D'R \sqrt{\frac{C}{L}}$</td>
<td>$\frac{D^2R}{DL}$</td>
</tr>
</tbody>
</table>

**Fig. 8.27**  
Bode plot of the line-to-output transfer function $G_{Vg}$, buck-boost converter example. Phase reversal due to output voltage inversion not included.

![Bode plot](image-url)