Linear Time Invariant Systems: Continuous Time Convolution
Today’s Topics: LTI Systems

• Continuous Time Response
• Motivation for Continuous Time Impulse Response Representation
• Continuous Time Convolution
• Examples
Notes

• Two sums

\[ \sum_{k=0}^{N} k = ? \]

\[ \sum_{k=0}^{N} \alpha^k = ? \]
Continuous Time Response

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h_\tau[t] \, d\tau \]

- Need to measure \( h \) at each time
Continuous Time Convolution

- When system is LTI

\[
y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h[t - \tau] \, d\tau
\]

\[
y(t) = x(t) \ast h(t)
\]
To Motivate Impulse Response

- Delta function is not quite legitimate so begin by using rectangles with a free parameter
Sampling

\[ \hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta) \delta_{\Delta}(t - k\Delta) \Delta \]

• Staircase approximation
Sampling at a Single Point

- Use value at middle of sampling range
Sampling Function

\[ \delta_\Delta(t) = \begin{cases} 
\frac{1}{\Delta}, & 0 \leq t \leq \Delta \\
0, & \text{otherwise} 
\end{cases} \]

- Has normalization = 1 independent of parameter

\[ \int_{-\infty}^{\infty} \delta_\Delta(\tau) d\tau = 1 \]
Sampling at a Single Point

- In limit of delta going to zero, the range and the point coincide
\[ x(t) = \lim_{\Delta \to 0} \hat{x}(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)\,d\tau \]

- Will use to find impulse response
**Impulse Response**

- Can define a time dependent response

\[
\begin{align*}
\text{When } x(t) &= \delta_\Delta(t - k\Delta) \\
\text{Then } \hat{y}(t) &= \hat{h}_{k\Delta}(t)
\end{align*}
\]
System Response

- Successive impulse responses
System Response II

\[ \hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta) \delta_{\Delta}(t - k\Delta) \Delta \]
Impulse Response

\[ \hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta) \hat{h}_{k\Delta}(t) \Delta \]

- Results in
Impulse Response

• Taking a limit in Delta results in

\[ y(t) = \int_{-\infty}^{\infty} x(\tau)h_\tau(t) d\tau \]
LTI Impulse Response

- When the response is LTI

\[
y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau
\]

\[
y(t) = x(t) \ast h(t)
\]
Example 2.6

\[ y(t) = x(t) \ast h(t) \]

- With

\[ x(t) = \exp(-\alpha t)u(t) \]

\[ h(t) = u(t) \]
Example 2.6

- First question: using the \( u(t) \)'s, when do the two overlap?
Example 2.6

- Correct limits result in

\[ y(t) = \int_{\tau=0}^{t} x(\tau) h(t - \tau) \, d\tau \]

- When \( x \) is exponential, what is the integral? If \( h \) were the exponential...?
Example 2.6

- Result is

\[ y(t) = \frac{1}{a} (1 - e^{-at})u(t) \]
Example 2.7

\[ y(t) = x(t) \ast h(t) \]

With

\[ x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases} \]

\[ h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases} \]
**Example 2.7**

\[ x(t) = \begin{cases} 
1, & 0 < t < T \\
0, & \text{otherwise}
\end{cases} \]

\[ h(t) = \begin{cases} 
t, & 0 < t < 2T \\
0, & \text{otherwise}
\end{cases} \]

- Approach: 0) What does the answer look like?
- 1) find break points, 2) do integrals
Example 2.7

(a) $0 < t < T$

(b) $T < t < 2T$

(c) $2T < t < 3T$
Example 2.7

\[ y(t) = x(t) \star h(t) \]

- The Problem
Example 2.7

\[ y(t) = x(t) \ast h(t) \]

- Result
Example 2.8

\[ y(t) = x(t) * h(t) \]

- With

\[ x(t) = \exp(2t)u(-t) \]

\[ h(t) = u(t - 3) \]
Example 2.8

\[ x(t) = \exp(2t)u(-t) \]
\[ h(t) = u(t - 3) \]

- Approach: 0) Sketch the functions and then what the answer looks like? 1) find break points, 2) do integrals
Example 2.8

- Details in the book