ECEN 3300
Linear Systems
Class Meeting 16

Singularity Functions
Today’s Topics: Singularity Functions

- Singularity Functions
- Representations of Delta Functions
- Driving with Finite Impulses
- Delta Identities
- Derivatives of Delta Functions
Singularity Functions

\[
\delta(t) = \delta(t) \ast \delta(t)
\]

\[
f(t)\delta(t) = f(0) \ast \delta(t)
\]

\[
\frac{d^2y(t)}{dt^2} = y(t) \ast u_1(t) \ast u_1(t)
\]

- Singularity functions arise from using idealizations such as steps and, especially, impulses
Sifting for Identities

\[ x(t) = x(t) \ast \delta(t) \]

- **When**

\[ x(t) = \delta(t) \]

\[ \delta(t) = \delta(t) \ast \delta(t) \]

- **Can this be right?**
Representations for Delta

\[ \delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t) \]

\[ \delta_\Delta(t) = \frac{1}{\Delta} \left( u(t) - u(t - \tau) \right) \]

\[ r_\Delta(t) = \delta_\Delta(t) \ast \delta_\Delta(t) \]

- What is the limit of this?
- What does it look like?
Representations for Delta II

- What is the area under the curve?
- What is the height and where?
Representations for Delta III

\[ \delta(t) = \sum_{k=0}^{\infty} a_k \delta(t) \ast^k \delta(t) \]

\[ = a_0 \delta(t) + a_1 \delta(t) \ast \delta(t) + a_2 \delta(t) \ast \delta(t) \ast \delta(t) + \ldots \]

• Can you think of some functions that reduce to \( \delta(t) \)?

• Why would it be useful to know some?
Pulsed Drivers: Example 2.16

\[
\frac{dy(t)}{dt} + 2y(t) = x(t)
\]

\[
x(t) = \left\{ \begin{array}{c}
\delta_{\triangle}(t) \\
r_{\triangle}(t) \\
\delta_{\triangle}(t) \ast r_{\triangle}(t) \\
r_{\triangle}(t) \ast r_{\triangle}(t)
\end{array} \right.
\]

How to find these? What do they look like?
Example 2.16
Pictures

Do these look familiar?
Example 2.16

Do these look familiar?
Defining Delta without Infinity

\[ x(t) = x(t) \ast \delta(t) \]

*If* \( x(t) = 1 \)

*Then* \( \int_{-\infty}^{\infty} \delta(\tau) d\tau = 1 \)

Other properties of generalized functions follow from the sifting property
Using Delta to Sample

\[ x(t) = x(t) \ast \delta(t) \]

If \( x(t) = g(-t) \)

Then \( \int_{-\infty}^{\infty} g(\tau)\delta(\tau)d\tau = g(0) \)

Other properties of generalized functions follow from the sifting property
Using Delta to Fix Argument

From

\[ x(0) = \int_{-\infty}^{\infty} x(\tau) \delta(\tau) d\tau \]

Can show

\[ f(t) \delta(t) = f(0) \delta(t) \]

How?
Derivatives of Delta Functions

What happens then when

\[ y(t) = x(t) \ast h(t) \]

What circuit realizes this?
Derivatives of Delta Functions II

\[
\frac{dx(t)}{dt} = x(t) * h(t)
\]

with solution

\[
h(t) = u_1(t)
\]

- Can we measure this \( h(t) \)?
- What does it look like?
A Picture of $u_1$

- Are there more?
Higher Derivatives

\[ y(t) = x(t) \ast u_1(t) \ast u_1(t) \]

with alternate representation

\[ h(t) = u_1(t) \ast u_1(t) = u_2(t) \]

• Can we measure this \( h(t) \)?
• What does it look like?
Still Higher Derivatives

What are

\[ y(t) = x(t) \ast u_n(t) \]

\[ y(t) = \sum_{k=0}^{\infty} a_k x(t) \ast u_k(t) \]

Can we generate anything we want?
Negative Derivatives

What happens then when

\[ y(t) = \int_0^t x(\tau) \, d\tau \]

What circuit realizes this?
Negative Derivatives II

with solution

\[ h(t) = u(t) = u_{-1}(t) \]

- Can we measure this \( h(t) \)?
- What does it look like?
More Negative

\[ y(t) = x(t) \ast u_{-1}(t) \ast u_{-1}(t) \]

with alternate representation

\[ h(t) = u_{-1}(t) \ast u_{-1}(t) = u_{-2}(t) \]

• What does this look like?
Picturing an Integral Operator

• Can we define $u_{-n}(t)$?
• Can we draw $u_{-n}(t)$?
Still More Negative

What are?

\[ y(t) = x(t) \ast u_{-n}(t) \]

\[ y(t) = u_k(t) \ast u_r(t) \]