ECEN 3300
Linear Systems
Class Meeting 20

Properties of Fourier Series
Today’s Topics: Properties of FS

• Convergence of Fourier Series
  – Dirichlet Conditions
  – Gibb’s phenomena

• Properties of Series
  – Linearity
  – Time shift/reversal
  – Time scaling
  – Multiplication
  – Conjugate symmetry
  – Parseval symmetry
Fourier Series

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k \exp(jk\omega_0 t) \]

\[ a_k = \frac{1}{T} \int_{0}^{T} x(t) \exp(-jk\omega_0 t) dt \]

\[ \omega_0 = \frac{2\pi}{T} \]

• Definition of Fourier Series Coefficients
Convergence of Fourier Series

- Dirichlet Conditions
  1. Absolute integrability
  2. Bounded variation
  3. Finite number of discontinuities

- What do they mean and do they have mathematical statements?
Example 3.5 (Revisited)

• Find the series representation for

• What does a plot of the coefficients look like?
Coefficients for Example 3.5

- Why are there so many non-zero coefficients?
Gibb’s Phenomena

- Does not converge uniformly near the edge
Properties of Fourier Series

- **Linearity**

\[ x(t) \leftrightarrow a_k \]
\[ y(t) \leftrightarrow b_k \]
\[ x(t) + y(t) \leftrightarrow a_k + b_k \]

- Fourier series (transform) is linear so superposition applies

- How to show?
Time Shift

\[ x(t) \leftrightarrow a_k \]

\[ x(t - t_0) \leftrightarrow a_k \exp(-jk\omega_0 t_0) \]

• How to show?
Time Reversal

\[ x(t) \leftrightarrow a_k \]

\[ x(t) \rightarrow x(-t) \]

\[ a_k \rightarrow a_{-k} \]

• How to show?
Time Scaling

\[ x(t) = \sum_{-\infty}^{\infty} a_k \exp(jk\omega_0 t) \]

\[ x(\alpha t) = \sum_{-\infty}^{\infty} a_k \exp(jk(\alpha\omega_0)t) \]

• Have the a’s changed?
• What has changed?
Time Scaling II

\[ a_k = \frac{1}{T} \int_{-\infty}^{\infty} (x(\tau) \exp(-jk\omega_0 \tau)) d\tau \]

\[ b_k = \frac{1}{\alpha T} \int_{0}^{\alpha T} (x(\alpha \tau) \exp(-jk\omega_0 \alpha \tau)) d(\alpha \tau) = a_k \]

- The a’s haven’t changed?
- The period has changed?
Multiplication

\[ x(t) \leftrightarrow a_k \]

\[ y(t) \leftrightarrow b_k \]

\[ x(t)y(t) \leftrightarrow \sum_{i=-\infty}^{\infty} a_{k-i}b_i = \sum_{i=-\infty}^{\infty} a_i b_{k-i} \]

• Relation between times and convolution

• Leave the details to the reader
Conjugate Symmetry

\[ x(t) \leftrightarrow a_k \]

\[ x^*(t) \leftrightarrow a^*_{-k} \]

- How to show?
- What does this say about representations of real signals?
- Imaginary signals?
Parseval’s Relation

\[ x(t) \leftrightarrow a_k \]

\[ \frac{1}{T} \int_{0}^{T} |x(t)|^2 dt = \sum_{-\infty}^{\infty} |a_k|^2 \]

- How to show?
- Why is this useful?
Example 3.6

• Find the Fourier series for

• Relates back to problem 3.5
Example 3.5 (Revisited)

- Find the series representation for

- What does a plot of the coefficients look like?
Coefficients for Example 3.5

- Why are there so many non-zero coefficients?
Example 3.6

• Find the Fourier series for

• Relates back to problem 3.5 – How?
Example 3.6 Solutions

- Note that

\[ g(t) = x(t - 1) - \frac{1}{2} \]
Example 3.7

- Find the $a$ in $x(t) \leftrightarrow a_k$
Example 3.7

• Note that

• Is the derivative of

• And integrate by parts
Example 3.8

\[ x(t) = \sum_{-\infty}^{\infty} \delta(t - kT) \]

- Find the \( a \) in

\[ x(t) \leftrightarrow a_k \]

- Apply to a train of square pulses width \( 2T \)
Example 3.9

\( x(t) \leftrightarrow a_k \)

- \( x(t) \) is real
- \( x(t) \) has period \( T=4 \)

\[ a_k = 0 \text{ for } |k| > 1 \]

\( y(t) \leftrightarrow b_k \) is odd with

\[ b_k = \exp(-j\pi k/2)a_k \text{ is odd} \]

\[ \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt = \frac{1}{2} \]

- What is \( x(t) \)?