ECEN 3300
Linear Systems
Class Meeting 21

Fourier Series of Discrete Signals
Today’s Topics: Properties of FS

• Fourier Series in General
  – Continuous
  – Discrete

• Discrete Fourier Series
  – Definitions
  – Examples
Continuous Fourier Series

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k \exp(jk\omega_0 t) \]

\[ a_k = \frac{1}{T} \int_{0}^{T} x(t) \exp(-jk\omega_0 t) dt \]

\[ \omega_0 = \frac{2\pi}{T} \]

• Definition of Fourier Series Coefficients
Why Fourier Representation?

- Simpler Representation
- Blocks are a Functional Representation

\[
x(t) \rightarrow h(t) \rightarrow h(t) \ast x(t)
\]

\[
a_k \rightarrow H(k) \rightarrow H(k\omega_0) a_k
\]

\[
H(s) = \mathcal{L} \{h(t)\}
\]
Discrete Fourier Series

\[ x[n] = \sum_{k=\langle N \rangle} a_k \exp(jk\omega_0 n) \]

\[ a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] \exp(-jk\omega_0 n) \]

\[ \omega_0 = \frac{2\pi}{N} \]

• Definition of Discrete Fourier Series
Why Discrete Fourier?

- Simpler Representation
- Blocks are a Functional Representation
Discrete Fourier Series

• Periodic function

\[ x[n] = x[n + N] \]

• Periodic basis set

\[ \phi_k[n] = \exp(jk\omega_0 n) \]

\[ \omega_0 = \frac{2\pi}{N} \]

• To expand in the basis set
Discrete Fourier Expansion

\[ x[n] = \sum_{k} a_k \phi_k[n] \]

- Have that

\[ \phi_k[n] = \phi_{k+N}[n] \quad \omega_0 = \frac{2\pi}{N} \]

- So that

\[ x[n] = \sum_{k=\langle N \rangle} a_k \phi_k[n] \]

- The a’s repeat with the phi’s
Using Identities

\[ x[0] = \sum_{k=\langle N \rangle} a_k \quad x[1] = \sum_{k=\langle N \rangle} a_k \exp[j2\pi k/N] \]

\[ x[N - 1] = \sum_{k=\langle N \rangle} a_k \exp[j2\pi (N - 1)/N] \]

• and

\[ \sum_{k=\langle N \rangle} \exp[jk(2\pi/N)n] = \begin{cases} N, & k = 0, \pm N, 2 \pm N, \ldots \\ 0, & \text{otherwise} \end{cases} \]
Discrete Fourier Series

\[ x[n] = \sum_{k=\langle N \rangle} a_k \exp(jk\omega_0 n) \]

\[ a_k = \frac{1}{\langle N \rangle} \sum_{n=\langle N \rangle} x[n] \exp(-jk\omega_0 n) \]

\[ \omega_0 = \frac{2\pi}{N} \]

• Definition of Discrete Fourier Series
Example 3.10a

\[ x[n] = \sin \omega_0 n \]

\[ \omega_0 = \frac{2\pi}{N} \]

• Find the series representation
Example 3.10a

\[ x[n] = \sin \omega_0 n \]

\[ = \frac{1}{2j} \exp(j\frac{2\pi}{N}n) - \frac{1}{2j} \exp(-j\frac{2\pi}{N}n) \]

\[ = a_1 \exp(j\frac{2\pi}{N}n) - a_{-1} \exp(-j\frac{2\pi}{N}n) \]

- Only two non-zero coefficients
Example 3.10a

\[ x[n] = \sin \omega_0 n \]

\[ = \frac{1}{2j} \exp(j(2\pi/5)n) - \frac{1}{2j} \exp(-j(2\pi/5)n) \]

• Only two non-zero coefficients
Example 3.10b

\[ x[n] = \sin M\omega_0 n \]

\[ \omega_0 = \frac{2\pi}{N} \]

• Find the series representation
Example 3.10b

\[ x[n] = \sin \omega_0 n \]

\[ = \frac{1}{2j} \exp(jM(2\pi/N)n) - \frac{1}{2j} \exp(-jM(2\pi/N)n) \]

\[ = a_M \exp(jM(2\pi/N)n) - a_{-M} \exp(-jM(2\pi/N)n) \]

• Only two non-zero coefficients
Example 3.10b

\[ x[n] = \sin \omega_0 n \]

\[ = \frac{1}{2j} \exp(j3(2\pi/5)n) - \frac{1}{2j} \exp(-j3(2\pi/5)n) \]

- Only two non-zero coefficients
Example 3.11

\[ x[n] = 1 + \sin(\omega_0 n) + 3 \cos(\omega_0 n) + \cos\left(2\omega_0 n + \frac{\pi}{2}\right) \]

- How to find the expansion?
- What does it look like?
Example 3.11

\[ x[n] = 1 + \sin (\omega_0 n) + \cos (\omega_0 n) + \sin (2\omega_0 n) \]

- The plots show the real, the imaginary and the magnitude
Example 3.12

\[ a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} \exp(-j k \omega_0 n) \]

• How to find the expansion?
• What to do when given coefficients instead of function?
Remembering some facts

• Infinite sums are hard to do unless you already know the answer
• The previous example, especially examples 3.5 – 3.9, have introduced practically all of the infinite sums that are doable.
Continuous Coefficients

\[ a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} \exp(-jk\omega_0 n) \]

- How to find the expansion?
- What to do when given coefficients instead of function?
Example 3.5

• Find the series representation for

• What do the coefficients look like?
Example 3.5

\[ a_k = \frac{\sin (k \omega_0 T_1)}{k \pi} \]
Coefficients for Example 3.5

- Rectangles transform to sinc’s
Related to Example 3.8

- Pulses transform to constants

\[ a_k = \frac{1}{T} \]
Example 3.12

\[ a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} \exp(-jk\omega_0 n) \]

- How to find the expansion?
- What to do when given coefficients instead of function?
Example 3.12

\[ a_k = \begin{cases} 
\frac{1}{N} \frac{\sin[2\pi k(N_1 + 1/2)/N]}{\sin \pi k/N} & \text{if } k \neq 0, \pm N, \ldots \\
\frac{2N_1 + 1}{N} & \text{if } k = 0, \pm N, \ldots
\end{cases} \]