ECEN 3300
Linear Systems
Class Meeting 31

Discrete Fourier Transform II
Today’s Topics

• The DFT of a Periodic Signal
  – Example 5.5
  – Example 5.6

• The FTs and DFTs we know
  – Exponential
  – Delta
  – Rect

• Properties of the DFT
The DFT of a Periodic Signal

- The Discrete Fourier Transform
- The DFT of a Delta Function
- The Fourier Series
- The Discrete Fourier Series
- The DFS of a delta Function
- The DFT of a DFS
Discrete Fourier Transform

\[ x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) \exp(j\omega n) d\omega \]

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \exp(-j\omega n) \]

- The complex exponential is periodic in omega
- Even if the single is aperiodic, the transform is periodic with period 2pi
DFT of a Delta Function

\[ x[n] = \delta[n] \quad \text{aperiodic} \]

- What does this look like

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \exp(-j\omega n) \]

- What does the transform look like?
- Is it periodic?
Fourier Series

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k \exp(jk\omega_0 t) \]

\[ a_k = \frac{1}{T} \int_{0}^{T} x(t) \exp(-jk\omega_0 t) \, dt \]

\[ \omega_0 = \frac{2\pi}{T} \]

- Fourier Series Expresses a Periodic Continuous Time Function in terms of discrete set of coefficients
Discrete Fourier Series

\[ T = N dt \quad t = ndt/k \quad dt \to \Delta t \to 1 \]

\[ x[n] = \sum_{k=\langle N \rangle} a_k \exp(jk\omega_0 n) \]

\[ a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] \exp(-jk\omega_0 n) \]

- Both the number of samples and the number of coefficients are discrete, but here also finite
DFS of a delta Function

\[ x[n] = \delta[n] \quad \text{with period } N \]

\[ a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] \exp(-j k \omega_0 n) \]

\[ \omega_0 = \frac{2\pi}{N} \]

- How do the a’s depend on N? On the k’s?
Delta function Definitions

\[ 2\pi \delta(\omega) = \lim_{N \to \infty} \sum_{n=-N}^{N} e^{-j\omega n} \]

\[ 2\pi \delta(\omega) = \lim_{T \to \infty} \int_{-T}^{T} e^{-j\omega t} dt \]

• Not too hard to show
A DFT of a DFS

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \exp(-j\omega n) \]

\[ x[n] = \sum_{k=\langle N \rangle} a_k \exp(jk\omega_0 n) \]

• What does this look like?
A DFT of a DFS - Result

\[ X(e^{j\omega}) = \sum_{k=\langle N\rangle} 2\pi a_k \delta \left( \omega - \frac{2\pi k}{N} \right) \]

- The transform is continuous and periodic in 2 pi
Example 5.5

\[ x[n] = \cos(\omega_0 n) \]

- Find and Sketch

\[ X(e^{j\omega}) = ? \]

- How about similar function like \( \sin \), \( \cos^2 \), delta, sums of delta’s, rect, …
Example 5.5 Solution

$$x[n] = \cos \omega_0 n$$

- The spectrum repeats
Example 5.6

\[ x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN] \]

- Find the Fourier coefficients
- Sketch the DFT spectrum
Example 5.6 Solution

\[ a_k = \frac{1}{N} \quad X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta \left( \omega - \frac{2\pi k}{N} \right) \]
Fourier Transform

\[ x(t) = \frac{1}{2\pi} \int_{k=-\infty}^{\infty} X(j\omega) \exp(j\omega t) d\omega \]

\[ X(j\omega) = \int_{-\infty}^{\infty} x(t) \exp(-j\omega t) dt \]

• There are only so many pairs we know
Transforms We Can Do

\[ e^{-at}u(t) \leftrightarrow \frac{1}{j\omega + a} \]

\[ \delta(t) \leftrightarrow 1 \]

\[ \text{rect}(t) \leftrightarrow \frac{\sin(\omega/2)}{\omega/2} \]

• Here rect is defined to have unity area
What we can do to transforms we know

$$e^{-at}u(t) \leftrightarrow \frac{1}{j\omega + a}$$

$$\delta(t) \leftrightarrow 1$$

$$\text{rect}(t) \leftrightarrow \frac{\sin(\omega/2)}{\omega/2}$$

- Conjugate Symmetry
- Duality
- Time (frequency) shift
- Differentiation
- Parseval