ECEN 3300
Linear Systems
Class Meeting 35

Difference Equations
Today’s Topics

• First Order Difference Equations
  – Examples

• Constant Coefficient Difference Equations
  – Examples 5.18
  – Example 5.19
  – Example 5.20

• Second Order Difference Equations
  – Examples
First Order Equations

\[ \frac{dy(t)}{dt} + \alpha y(t) = x(t) \]

\[ y[n] - ay[n - 1] = x[n] \]

• Using backward difference, one can find a correspondence between the equations giving a relation between a and alpha
First Order LTI Differential Equations

\[ \frac{dy(t)}{dt} + \alpha y(t) = x(t) \]

- This equation can be solved in general without knowing \( x(t) \)
LTI Integrating Factor

\[ \frac{dy(t)}{dt} + \alpha y(t) = x(t) \]

\[ x(t)|_{t<0} = 0 \]

\[ y(t)|_{t<0} = 0 \]

\[ y(t) = e^{\alpha t} \int_{0}^{t} x(t')e^{-\alpha t'}dt' \]

- Solution for all causal \( x(t) \)
Backward Recursion for LTI Difference

\[ y[n] - ay[n - 1] = x[n] \]
\[ x[n]|_{n<0} = 0 \]
\[ y[n]|_{n<0} = 0 \]

- There is no way to find an integrating factor
- There is a homogeneous solution but the particular solution is not well defined
- Can solve by backward recursion
Backward Recursion for LTI Difference

\[ y[n] - ay[n - 1] = x[n] \]

\[ x[n] |_{n<0} = 0 \]

\[ y[n] |_{n<0} = 0 \]

\[ y[n] = x[n] + ay[n - 1] \]

• Solution for all causal \( x[n] \)
Impulse and Step Response

\[
\frac{dy(t)}{dt} + \alpha y(t) = x(t) \quad y(0) = 0
\]

- **Impulse**
  \[x(t) = \delta(t)\]

- **Step**
  \[x(t) = u(t)\]

- **Ramp**
  \[x(t) = tu(t)\]
Impulse and Step Response

\[ y[n] - ay[n - 1] = x[n] \quad y[-1] = 0 \]

- **Impulse**
  \[ x[n] = \delta[n] \]

- **Step**
  \[ x[n] = u[n] - u[n - M] \]

- **Ramp**
  \[ x[n] = nu[n] \]
Frequency Domain

\[
\frac{dy(t)}{dt} + \alpha y(t) = x(t)
\]

\[y(t) = h(t) \ast x(t)\]

- Can find \( h(t) \) from integrating factor solution

\[Y(j\omega) = H(j\omega)X(j\omega)\]
Frequency Domain

\[
\frac{dy(t)}{dt} + \alpha y(t) = x(t)
\]

\[y(t) \leftrightarrow Y(j\omega)\] \[x(t) \leftrightarrow X(j\omega)\]

\[
\frac{dy(t)}{dt} \leftrightarrow j\omega Y(j\omega)
\]

- Can find $H$ from the transforms
Frequency Domain

\[ y[n] - ay[n - 1] = x[n] \]

\[ y(t) = h(t) \ast x(t) \]

- Can only find \( h(t) \) from backward recursion solution for some \( x[n] \) – impulse, step, etc.

\[ Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) \]
Frequency Domain

\[ y[n] - ay[n-1] = x[n] \]

\[ y[n] \leftrightarrow Y(e^{j\omega}) \quad x[n] \leftrightarrow X(e^{j\omega}) \]

\[ y[n - 1] \leftrightarrow e^{-j\omega}Y(j\omega) \]

- Can find H from the transforms
Difference Equations

- A general discrete LTI systems can be described by

\[ \sum_{k=0}^{N} a_k y[n - k] = \sum_{k=0}^{M} b_k x[n - k] \]
Impulse Response

• A general discrete LTI systems has an impulse response defined by

\[ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \]

• Transforming both sides of the defining difference equation yields
Impulse Response

• Using the defining difference equations then gives

\[
H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}
\]

\[
H(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} / \sum_{k=0}^{N} a_k e^{-jk\omega}
\]
Example 5.18

\[ y[n] - ay[n - 1] = x[n] \]

• Find the solution of the homogenous equation given a \( y[0] \)
• Is the system causal?
• Find the general solution given causal \( x[n] \)
• Is the system finite impulse response? Is the system recursive? Are FIR and recursive related?
• Find the impulse response \( h[n] \)
Example 5.18 Frequency Domain

\[ y[n] - ay[n - 1] = x[n] \]

- Find the transfer function
- Invert the transfer function to find \( h[n] \)
- Find the transform of the step response
- Invert the step response to find \( s[n] \)
- Find the transform of the ramp response
- Invert ...
Example 5.18 Frequency Domain

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