ECEN 3300
Linear Systems
Class Meeting 36

Difference Equations II
Today’s Topics

- Review First Order Equations
- Second Order Equations
  - Differential Equations
  - Difference Equations
- Constant Coefficient Difference Equations
  - Example 5.19
  - Example 5.20
First Order Equations

\[
\frac{dy(t)}{dt} + \alpha y(t) = x(t)
\]

\[
y[n] - ay[n - 1] = bx[n]
\]

- Using backward difference, one can find a correspondence between the equations giving a relation between a and alpha

\[
y[n] - \frac{y[n - 1]}{1 + \alpha} = \frac{x[n]}{1 + \alpha}
\]
First Order LTI Differential Equations

\[ \frac{dy(t)}{dt} + \alpha y(t) = \alpha x(t) \]

- This equation can be solved in general using an integrating factor that is equivalent to convolution with \( h(t) \)
- We can find the frequency response and impulse response
First Order LTI Differential Equations

\[
\frac{dy(t)}{dt} + \alpha y(t) = \alpha x(t)
\]

\[
h(t) \leftrightarrow H(j\omega)
\]

\[
e^{-\alpha t} u(t) \leftrightarrow \frac{1}{j\omega + \alpha}
\]

- This equation can be solved in general using an integrating factor that is equivalent to convolution with \(h(t)\)
First Order LTI Difference Equations

\[ y[n] - ay[n - 1] = x[n] \]

• This equation can be solved in general using by backward recursion that is equivalent to convolution with \( h[n] \)

\[ y[n] - |a| \cos \theta y[n - 1] = x[n] \]

• Sometimes write \( a \) as amplitude and phase – phase determines oscillation
First Order LTI Difference Equations

\[ y[n] - ay[n - 1] = x[n] \]

- Can find the frequency response and impulse response in a straightforward manner
First Order LTI Difference Equations

\[ y[n] - ay[n - 1] = x[n] \]

\[ h[n] \leftrightarrow H(e^{j\omega}) \]

\[ a^n u[n] \leftrightarrow \frac{1}{1 - ae^{-j\omega}} \]

- Behaviors of \( h[n] \) and \( H \) can be striking depending on phase of \( a \)
First Order LTI Difference Equations

• Impulse response $h[n]$
Frequency Response LTI Difference Equations

\[ H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} \]

\[ |H(e^{j\omega})| = \frac{1}{1 + a^2 + 2a \cos \omega} \]

- Behavior of \( H \) can be striking depending on phase of \( a \)
First Order LTI Difference Equations

- Frequency response $H$
Second Order LTI Differential Equations

\[
\frac{d^2 y(t)}{dt^2} + 2\zeta \omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)
\]

\[h(t) \leftrightarrow H(j\omega)\]

• This equation can be solved in general by convolving with \( h(t) \)
Second Order LTI Differential Equations

\[
\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)
\]

\[h(t) \leftrightarrow H(j\omega)\]

\[(A_1 e^{j\omega t} + A_2 e^{-j\omega t}) u(t) \leftrightarrow \frac{1}{(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1}\]

• The solution has over-damped, critically damped or under-damped behaviors depending on the roots
Second Order LTI Differential Equations

\[ j(\omega_{\pm}/\omega_n) = -\zeta \pm \sqrt{\zeta^2 - \omega_n^2} \]

- Ringing or non-ringing is important
Second Order LTI Equations

\[
\frac{d^2y(t)}{dt^2} + 2\zeta \omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)
\]

\[
y[n] - ay[n - 1] + by[n - 2] = cx(t)
\]

- Can relate \(a\), \(b\) and \(c\) to zeta and omega by using backward differences to find
Second Order LTI Equations

\[ \frac{d^2y(t)}{dt^2} + 2\zeta \omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t) \]

\[ y[n] - 2 \left( \frac{1 - \zeta \omega_n}{1 + \omega_n^2} \right) y[n - 1] + \frac{1}{1 + \omega_n^2} y[n - 2] = \frac{\omega_n^2}{1 + \omega_n^2} x(t) \]

- The \( y[n-1] \) term can be negative (damped) or positive oscillatory
- Often write in terms of \( r \) and \( \theta \)
Second Order LTI Difference Equations

\[ y[n] - 2r \cos \theta y[n - 1] + r^2 y[n - 2] = x[n] \]

\[ h[n] \leftrightarrow H(e^{j\omega}) \]

\[ r^n \frac{\sin(n + 1)\theta}{\sin \theta} u[n] \leftrightarrow \frac{1}{1 - 2r \cos \theta e^{-j\omega} + r^2 e^{-2j\omega}} \]

- This equation can be solved by convolving with \( h[n] \)
Second Order LTI Difference Equations

- Oscillation determined by sign of $a$
Example 5.19 – Second Order LTI

\[ y[n] - \frac{3}{4}y[n - 1] + \frac{1}{8}y[n] = 2x[n] \]

- Is the system over or under-damped?
- Find the frequency response
- Find the impulse response
- Find the step response
Example 5.19 – Second Order LTI

\[ y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n] = 2x[n] \]

\[ H(e^{j\omega}) = \frac{1}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}} \]

\[ H(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega} \frac{1}{1 - \frac{1}{4}e^{-2j\omega}}} \]

\[ h[n] = A \left(\frac{1}{4}\right)^n u[n] - B \left(\frac{1}{8}\right)^n u[n] \]

• Highly damped behavior
Example 5.20 – Second Order LTI

\[ y[n] - \frac{3}{4}y[n - 1] + \frac{1}{8}y[n] = 2x[n] \]

\[ x[n] = \left(\frac{1}{4}\right)^n u[n] \]

- Find \( Y \)
- Find the form of the impulse response
Example 5.20 – Second Order LTI

\[ Y(e^{j\omega}) = \frac{1}{1 - 2e^{-j\omega}} \left( \frac{1}{1 - \frac{1}{4}e^{-2j\omega}} \right)^2 \]

\[ y[n] = -A \left( \frac{1}{4} \right)^n u[n] - B(n + 1) \left( \frac{1}{4} \right)^n u[n] + C \left( \frac{1}{2} \right)^n u[n] \]

- Under-damped with no oscillation
Higher Order Difference Equations

- A general discrete LTI systems can be described by

\[
\sum_{k=0}^{N} a_k y[n - k] = \sum_{k=0}^{M} b_k x[n - k]
\]
Impulse Response

• A general discrete LTI systems has an impulse response defined by

\[ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \]

• Transforming both sides of the defining difference equation yields
Impulse Response

- Using the defining difference equations then gives

\[ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \]

\[ H(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} \]

\[ \sum_{k=0}^{N} a_k e^{-jk\omega} \]