10

Time-Invariant Electric
Current in Solid and
Liquid Conductors

10.1 Introduction

The term *time-invariant electric current* implies a steady, time-constant motion of a very large number of small charged particles. The term *current* is used because this motion is somewhat similar to the motion of a fluid. A typical example is the steady motion of free electrons inside a metallic conductor, but there are other types of time-invariant currents as well. What causes organized motion of large numbers of electrons (or other charges)? The answer is an electric field, which unlike in the electrostatic case, does exist inside current-carrying conductors. Time-invariant currents are frequently also called *direct currents*, abbreviated *dc*. A domain in which currents exist is known as the *current field*.

Inside a metallic conductor with no electric field present, a free electron (or any other type of free charge) moves chaotically in all directions, like a gas molecule. If there is an electric field inside the conductor, the electrons (negative charges) are accelerated in the direction opposite to that of the local vector $E$. This accelerated motion lasts until the electron collides with an atom. We can imagine that the electron
then stops, transfers the acquired kinetic energy to the atom, is again accelerated in the opposite $E$, and so on. So the electrons acquire an average "drift" velocity under the influence of the field, and the result of this organized motion is an electric current.

There are three important consequences of this fact:

1. In solid and liquid conductors, where the average path between two collisions is very short, the drift velocity is in the direction of the force, i.e., the charges follow the lines of vector $E$.

2. Charges constantly lose the acquired kinetic energy to the atoms they collide with. This results in a more vigorous vibration of the atoms, i.e., a higher temperature of the conductor. This means that in the case of an electric current in conductors, the energy of the electric field is constantly converted into heat. This heat is known as Joule's heat. It is also frequently called Joule's losses because it represents a loss of electric energy.

3. In the steady time-invariant state, the motion of electric charges is time-invariant. The electric field driving the charges must in turn be time-invariant, and is therefore due to a time-constant distribution of charges. Such an electric field is identical to the electrostatic field of charges distributed in the same manner. This is a conclusion of extreme importance. All the concepts we derived for the electrostatic field (scalar potential, voltage, etc.) are valid for time-invariant currents.

Liquid conductors have pairs of positive and negative ions, which move in opposite directions under the influence of the electric field. The electric current in liquid conductors is therefore made of two streams of charged particles moving in opposite directions, but we have the same mechanism and the same effect of energy loss (Joule's heat) of current flow as in the case of a solid conductor. There is an additional effect, however, known as electrolysis—chemical changes in any liquid conductor that always accompany electric current.

In a class of materials called semiconductors, there are two types of charge carriers—negatively charged electrons and positively charged holes. In this case, the electric field is due to both types of charges and depends very much on their concentrations.

In gases, electric current is also due to moving ions, but the average path between two collisions is much longer than for solid and liquid conductors. The mechanism of current flow is therefore quite different.

In solid and liquid conductors the number of charges taking part in an electric current is extremely large. To understand this, recall that a solid or liquid contains on the order of $10^{28}$ atoms per cubic meter. It is not easy to understand these huge numbers. Perhaps it would help if we consider a volume of about $(0.1 \text{ mm})^3$ (a cube 0.1 mm on each side), which is barely visible by the naked eye. This tiny volume contains about $10^{12}$ atoms, which is more than one hundred times the number of humans on our planet! It is evident from this example that the term "electric current" is indeed appropriate.

*Questions and problems:* Q10.1
10.2 Current Density and Current Intensity: Point Form of Ohm's and Joule's Laws

Electric current in conductors is described by two quantities. The current density vector, \( \mathbf{J} \), describes the organized motion of charged particles at a point. The current intensity is a scalar that describes this motion in an integral manner, through a surface.

Let a conductor have \( N \) free charges per unit volume, each carrying a charge \( Q \) and having an average (drift) velocity \( \mathbf{v} \) at a given point. The current density vector at this point is then defined as

\[
\mathbf{J} = NQ\mathbf{v} \quad \text{amperes per m}^2 \ (\text{A/m}^2).
\]

(Definition of current density for one kind of charge carriers)

Note that this definition implies that the current density vector of equal charges of opposite sign moving in opposite directions is the same. Of course, motion of different charges in opposite directions physically is different. However, experiments indicate that practically all effects (Joule’s heat, chemical effects, magnetic effects) of an electric current depend on the product \( Q\mathbf{v} \), so it is convenient to adopt this definition for the current density vector.

If there are several types of free charges inside a conductor, the current density is defined as a vector sum of the expression in Eq. (10.1). For example, let the current be due to the motion of free charge carriers of charges \( Q_1, Q_2, \ldots, Q_n \), moving with drift velocities \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \). Let there be \( N_1, N_2, \ldots, N_n \) of these charge carriers per unit volume, respectively. The current density vector is then given by

\[
\mathbf{J} = \sum_{k=1}^{n} N_kQ_k\mathbf{v}_k \quad \text{(A/m}^2\text{)}.
\]

(Definition of current density for several kinds of charge carriers)

The current intensity, \( I \), through a surface is defined as the total amount of charge that flows through the surface during a small time interval, divided by this time interval. In counting this charge, opposite charges moving in opposite directions are added together. Thus

\[
I = \frac{dQ_{\text{through } S}}{dt} \quad \text{(C/s = A)}.
\]

(Definition of current intensity through a surface)

This can also be expressed in terms of the current density vector, as follows.

Consider a surface element \( dS \) of the surface \( S \) in Fig. 10.1. Let the drift velocity of charges at \( dS \) be \( \mathbf{v} \), their charge \( Q \), and their number per unit volume \( N \). During the
time interval $dt$ the charges move by a distance $v \, dt$ in the direction of $v$. Therefore, the charge that crosses $dS$ in $dt$ is

$$dQ_{\text{through } dS \text{ during } dt} = dS \, v \, dt \, \cos \alpha \, NQ,$$  \hspace{1cm} (10.4)

where $\alpha$ is the angle between the velocity vector and the normal to the surface element. The total charge through $S$ during interval $dt$ is obtained as a sum (integral) of these elemental charges over the entire surface, and the current intensity is obtained by dividing this sum by $dt$. Noting that $dS \, v \, \cos \alpha \, NQ = J \, dS \, \cos \alpha = J \cdot dS$, we obtain

$$I = \int_S J \cdot dS \quad \text{(A).}$$  \hspace{1cm} (10.5)

(Definition of current intensity through a surface in terms of the current density vector)

The unit for current is an ampere (A), equal to a coulomb per second (C/s). The unit for current density is A/m².

We now know that electric current in a conductor is produced by an electric field. We also know that in solid and liquid conductors the vectors $J$ and $E$ are in the same direction. For most conductors, vector $J$ is a linear function of $E$,

$$J = \sigma E \quad [\sigma - \text{siemens per meter (S/m)}]. \quad \hspace{1cm} (10.6)$$

(Point (local) form of Ohm's law)

Conductors for which (10.6) is valid are called linear conductors. The constant $\sigma$ is known as the conductivity of the conductor. The unit for conductivity is siemens per meter (S/m).

The reciprocal value of $\sigma$ is designated by $\rho$ and is known as the resistivity. The unit for resistivity is ohm · meter ($\Omega \cdot m$). Equation (10.6) can be written in the form
\[ E = \rho J \quad [\rho \text{ - ohm} \cdot \text{meter} (\Omega \cdot \text{m})]. \tag{10.7} \]

(Point (local) form of Ohm's law)

Both Eqs. (10.6) and (10.7) are known as the point form of Ohm's law for linear conductors because they give a relationship between the two field quantities at every point inside a conductor.

For metallic conductors, conductivities range from about 10 MS/m (iron) to about 60 MS/m (silver). The conductivity of seawater is about 4 S/m, that of ground (soil) is between \(10^{-2}\) and \(10^{-4}\) S/m, and conductivities of good insulators are less than about \(10^{-12}\) S/m.

We have already explained from a physical standpoint that there is a permanent transformation of electric energy into heat in every current field. Let us now derive the expression, known as Joule's law in point form, for the volume density of power in this energy transformation.

Let there be \(N\) charge carriers \(Q\) in the conductor, and let their local drift velocity be \(v\). The electric force on each charge is \(QE\). The work done by the force when moving the charge during a time interval \(dt\) is equal to \(QE \cdot (v \, dt)\). The work done in moving all the \(N \, dv\) charges inside a small volume \(dv\) is therefore

\[
dA_{\text{el.,forces}} = QE \cdot (v \, dt)N \, dv = J \cdot E \, dv \, dt \quad (\text{J}). \tag{10.8}
\]

If we divide this expression by \(dv \, dt\), we get the desired power per unit volume (volume power density)—the electric power that is lost to heat:

\[
p_{J} = \frac{dP_{J}}{dv} = J \cdot E = \frac{J^2}{\sigma} = \sigma E^2 \quad \text{watts/m}^3 (W/m^3). \tag{10.9}
\]

(Joule's law in point form)

If we wish to determine the power of Joule's losses in a domain of space, we just have to integrate the expression in Eq. (10.9) over that domain:

\[
P_{J} = \int_{V} J \cdot E \, dv \quad \text{watts} (W). \tag{10.10}
\]

(Joule's losses in a domain of space)

**Example 10.1—Fuses.** Electrical devices are frequently protected from excessive currents by fuses, one type of which is sketched in Fig. 10.2. The fuse conductor is made to be much thinner than the circuit conductors elsewhere. For example, let the radius of the circuit conductor be \(n^2\) times that of the fuse. If a current of intensity \(I\) exists in the circuit, the volume density of Joule's losses in the thin conductor section is \(n^4\) larger than those in the other section. In the case of excessive current, therefore, the thin conductor section melts long before the normal section is heated up. When the fuse melts, it becomes an open circuit and does not allow any
further current to flow and possibly damage the device protected by the fuse. Usually the thin conductor is a metal with a conductivity smaller than that of the thick wire.

Questions and problems: Q10.2 and Q10.3, P10.1 to P10.7

10.3 Current-Continuity Equation and Kirchhoff's Current Law

Experiments tell us that electric charge cannot be created or destroyed. This is known as the law of conservation of electric charge. The continuity equation is the mathematical expression of this law. Its general form is valid for time-varying currents, but it can easily be specialized for time-invariant currents.

Consider a closed surface \( S \) in a current field. Let \( \mathbf{J} \) be the current density (a function of coordinates and, in the general case considered here, of time). The definition of current intensity applies to any surface, so it applies to a closed surface as well. The current intensity, \( i(t) \), through \( S \), with respect to the outward normal, is given by Eq. (10.3):

\[
i(t) = \frac{dq(t) \text{out of } S \text{ in } dt}{dt}.
\]  

According to the law of conservation of electric charge, if some amount of charge leaves a closed surface, the charge of opposite sign inside the surface must increase by the same amount. So we can write Eq. (10.11) as

\[
i(t) = -\frac{dq(t) \text{inside } S \text{ in } dt}{dt}.
\]  

The current intensity can also be written in the form of Eq. (10.5), and

\[
\frac{dq(t) \text{inside } S \text{ in } dt}{dt} = \frac{d}{dt} \int_{\mathbf{v}} \rho(t) d\mathbf{v},
\]  

where \( \mathbf{v} \) is the volume enclosed by \( S \). Recall that by convention we always adopt the outward unit vector normal to a closed surface. Thus Eq. (10.12) can be rewritten in
the form
\[ \oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{d}{dt} \int_V \rho(t) \, dv. \]  \hspace{1cm} (10.14)

(General form of the current continuity equation, where surface \( S \) may vary in time)

This is the current continuity equation. Note that we can imagine the surface \( S \) to change in time, in which case the form of the continuity equation in Eq. (10.14) must be used. This, however, is needed only in rare instances. If the surface \( S \) does not change in time, the time derivative acts on \( \rho \) only. Because \( \rho \) is a function of both time and space coordinates, the ordinary derivative needs to be replaced by a partial derivative, and we obtain a much more important form of the current continuity equation:

\[ \oint_S \mathbf{J} \cdot d\mathbf{S} = -\int_V \frac{\partial \rho(t)}{\partial t} \, dv. \]  \hspace{1cm} (10.15)

(Current continuity equation for a time-invariant surface)

Although the current continuity equation is not a field equation, it is of fundamental importance in the analysis of electromagnetic fields because only sources (charges and currents) satisfying this equation can be real sources of the field.

Now let the current field be constant in time, in which case the charge density is also constant in time. The partial derivative of \( \rho \) on the right side of Eq. (10.15) is then zero, and both forms of the current continuity equation reduce to

\[ \oint_S \mathbf{J} \cdot d\mathbf{S} = 0. \]  \hspace{1cm} (10.16)

(Generalized Kirchhoff’s current law)

This equation tells us that in time-constant current fields the amount of charge that flows into a closed surface is exactly the same as that which flows out of it. Equation (10.16) represents, in fact, the generalized form of the familiar Kirchhoff’s current law from circuit theory. Indeed, if a surface \( S \) encloses a node of a circuit, there are currents only through the circuit branches, and Eq. (10.16) becomes

\[ \sum_{k=1}^{n} I_k = 0. \]  \hspace{1cm} (10.17)

We know that Kirchhoff’s current law in this form is applied also to circuits with time-varying currents. Considering the preceding discussion, it should be clear that in such cases it is only approximate. (Can you explain why?)
Example 10.2—Continuity equation applied to a circuit node. Let the surface $S$ enclose a node with four wires (Fig. 10.3), with dc currents $I_1$, $I_2$, $I_3$, and $I_4$. The vector $\mathbf{J}$ is nonzero only over small areas of $S$ where the wires go through the surface. There, the flux of $\mathbf{J}$ is simply the current intensity in that wire, so that Eq. (10.16) yields $-I_1 - I_2 + I_3 + I_4 = 0$, which is what we would get if we simply applied Kirchhoff’s current law to the node. How are the signs of the currents determined and what do they correspond to in Eq. (10.16)?

Questions and problems: Q10.4 and Q10.5

10.4 Resistors: Ohm’s and Joule’s Laws

A resistor is a resistive body with two equipotential contacts. A resistor of general shape is shown in Fig. 10.4. Assume that the material of the resistor is linear. We know that the resistivity, $\rho$, for linear materials does not depend on the current density. Then the current density, $\mathbf{J}$, is proportional at all points of the resistor to the current intensity $I$ through its terminals. Therefore, the electric field vector in the
resistor material, $E = \rho J$, and the potential difference between its terminals are also proportional to the current intensity,

$$V_+ - V_- = RI \quad [R - \text{ohms} \ (\Omega)].$$

(10.18)

where $R$ is a constant. This equation is known as Ohm's law. Resistors for which this equation holds are called linear resistors. The constant $R$ is called the resistance of the resistor. In some instances it can be computed starting from the defining formula in Eq. (10.18), but it can always be measured. The unit for resistance is the ohm ($\Omega$).

The reciprocal of resistance is called the conductance, $G$. Its unit is called the siemens ($S$). In the United States, sometimes the mho (ohm backwards) is used instead, but this is not a legal SI unit and we will not use it.

**Example 10.3—Resistance of a straight wire segment.** As an example of calculating resistance, consider a straight wire of resistivity $\rho$, length $l$, and cross-sectional area $S$. Let the current intensity in the wire be $I$. The current density vector is parallel to the wire axis, and its magnitude is $j = I/S$. The electric field vector is therefore also parallel to the wire axis, and its magnitude is $E = \rho j = \rho I/S$. The potential difference between the ends of the wire segment is $V_1 - V_2 = EI = \rho ll/S$. So the resistance of the wire segment is

$$R = \frac{\rho}{S} \quad (\Omega).$$

(10.19)

Consider now a resistor of resistance $R$. Let the current intensity in the resistor be $I$, and the voltage between its terminals $V$. During a time interval $t$, a charge equal to $Q = It$ flows through the resistor. This charge is transported by electric forces from one end of the resistor to the other end. From the definition of voltage, the work done by electric forces is

$$A_{\text{el. forces}} = QV = VIt \quad (J).$$

(10.20)

Because of energy conservation, an energy equal in magnitude to this work is transformed into heat inside the resistor:

$$W = VIt = RI^2t = \frac{V^2}{R}t \quad (J).$$

(10.21)

Since the process of transformation of electric energy into heat is constant in time, the power of this transformation of energy is $W/t$, that is,

$$P = VI = RI^2 = \frac{V^2}{R} \quad (W).$$

(10.22)

(Joule's law)

This is the familiar Joule's law from circuit theory. It is named after the British physicist James Prescott Joule (1818–1889), who established this law experimentally.

**Questions and problems:** Q10.6 to Q10.9, P10.8 to P10.14
10.5 Electric Generators

We know that actual sources of the electric field are electric charges. We also know that we must remove some charges from a body, or put them on a body, in order to obtain excess electric charges on it. This obviously cannot be done by the electric forces themselves. Devices that do this must use some nonelectric energy to separate one type of charge from another. Such devices are known as electric generators.

An electric generator is sketched in Fig. 10.5. In a region of the generator there are nonelectric forces, known as impressed forces, that separate charges of different signs on the two generator terminals, or electrodes, denoted in the figure by "−" (negative) and "+" (positive). These forces can be diverse in nature. For example, in chemical batteries these are chemical forces; in thermocouples these are forces due to different mobilities of charge carriers in the two conductors that make the connection; in large rotating generators these are magnetic forces acting on charges inside conductors.

Impressed forces, by definition, act only on charges. Therefore they can always be represented as a product of the charge on which they act and a vector quantity that must have the dimension of the electric field strength:

\[ \mathbf{F}_{\text{impressed}} = Q \mathbf{E}_i. \]  

The vector \( \mathbf{E}_i \) is not necessarily an electric field strength (although it does have the same dimension and unit). It is known as the impressed electric field strength. The region of space it exists in is known as the impressed electric field. The impressed electric field is a concept used often in electromagnetic theory. For example, in the analysis of radar wave scattering from a radar target, the radar wave is considered to be a field impressed on the target.

The electromotive force, or emf, of a generator is defined as the work done by impressed forces in taking a unit charge through the generator, from its negative to

![Figure 10.5 Sketch of an electric generator](image-url)
its positive terminal:

\[ \mathcal{E} = \left\{ \int_{-}^{+} \mathbf{E}_i \cdot d\mathbf{l} \right\}_{\text{through the generator}} \quad \text{(definition of emf).} \quad (10.24) \]

By simple reasoning, the emf of a generator can be expressed in terms of the voltage between open-circuited generator terminals, as follows. Assume that the generator is open-circuited (no current is flowing through the generator). Then at all its points, the electric field strength due to separated charges and the impressed electric field strength must have equal magnitudes and opposite directions, i.e., \( \mathbf{E} = -\mathbf{E}_i \) (otherwise free charges in the conducting material of the generator would move). If we substitute this into Eq. (10.24) and exchange the integration limits, we get

\[ \mathcal{E} = \left\{ \int_{+}^{-} \mathbf{E} \cdot d\mathbf{l} \right\}_{\text{any path}} = V_+ - V_- . \quad (10.25) \]

The electric field strength \( \mathbf{E} \) is due to time-constant charges, i.e., it is an electrostatic field. The line integral can therefore be taken along any path, including one outside the generator (dashed line in Fig. 10.5). This means that we can measure the emf of a generator by a voltmeter with the voltmeter leads connected in any way to the terminals of an open-circuited generator. We will see that this is not the case for time-varying voltages.

Because the generator is always made of a material with nonzero resistivity, some energy is transformed into heat in the generator itself. Therefore, we describe a generator by its internal resistance also. This is simply the resistance of the generator in the absence of the impressed field.

*Questions and problems:* Q10.10 to Q10.12, P10.15 and P10.16

### 10.6 Boundary Conditions for Time-Invariant Currents

Current fields often exist in media of different conductivities separated by boundary surfaces. We now formulate boundary conditions for this case.

Consider the boundary surface sketched in Fig. 10.6. If we apply the continuity equation for time-invariant currents to the small, coinlike closed surface, we immediately see that the normal components of the current density vector in the two media must be the same:

\[ J_{1n} = J_{2n}, \quad \text{or} \quad \sigma_1 E_{1n} = \sigma_2 E_{2n}. \quad (10.26) \]

We know that the electric field in a current field has the same properties as the electrostatic field. Therefore the same boundary condition for the tangential component of the vector \( \mathbf{E} \) on a boundary applies:

\[ E_{1t} = E_{2t}, \quad \text{or} \quad \frac{J_{1t}}{\sigma_1} = \frac{J_{2t}}{\sigma_2}. \quad (10.27) \]
Example 10.4—Boundary conditions between a conductor and an insulator. If one of the two media in Fig. 10.6, say, medium 1, is an insulator (e.g., air), what are the expressions for the boundary conditions?

We know that there can be no current in the insulator. Hence from Eq. (10.26) we conclude that the normal component of the current density vector in a conductor adjacent to the insulator must be zero. Tangential components of the vector \( \mathbf{E} \) are the same in the two media, but of course \( j_{1\parallel} \) does not exist. Lines of vector \( \mathbf{J} \) in such a case are sketched in Fig. 10.7.

Questions and problems: Q10.13

10.7 Grounding Electrodes and an Image Method for Currents

Consider an electrode (a conducting body) of arbitrary shape buried in a poorly conducting medium (for example, soil) near the flat boundary surface between the poor conductor and some insulator (for example, air). Suppose that a current of intensity \( I \) is flowing from the electrode into the conducting medium, and is supplied from a distant current source through a thin insulated wire, as shown in Fig. 10.8. According to the boundary conditions, the current flow lines are tangential to the surface.
Imagine that the entire space is filled with the poor conductor. If an image electrode with the current of the same intensity flowing out of it is placed in the upper half of the space, as shown in the figure, the resulting current flow in both half spaces will be tangential to the plane of symmetry (the former boundary surface). Therefore, the current distribution in the lower half space is the same as in the actual case. Consequently, the influence of the boundary surface may be replaced by the image electrode, provided the current through the image electrode has the same intensity and direction as that through the real electrode. This method is very useful for analyzing current flow from electrodes buried under the surface of the earth. Such electrodes are often used for grounding purposes.

**Example 10.5—Hemispherical grounding electrode.** Suppose that a hemispherical electrode of very high conductivity is buried in poorly conducting soil of conductivity \( \sigma \), as shown in Fig. 10.9. We are interested in
The resistance of such a grounding system
The intensity of the electric field at all points on the earth’s surface if a current \( I \) flows
from the electrode into the earth
What happens if a person in noninsulating shoes approaches this grounding electrode

We use the image method. Let the current \( I \) flow out of the hemispherical electrode. If all
space is filled with earth, the image is another hemispherical electrode, so we get a spherical
electrode with current \( 2I \) in a homogeneous conducting medium. Due to symmetry, the current
from the spherical electrodes is radial. Consequently, the current from the original hemispherical
electrode is also radial. The current density at a distance \( r \) from the center of the hemisphere
is therefore

\[
J = \frac{2I}{4\pi r^2} = \frac{I}{2\pi r^2}.
\]

The magnitude of the electric field is

\[
E = \frac{J}{\sigma} = \frac{I}{2\pi \sigma r^2},
\]

so that the potential of the electrode with respect to a reference point at infinity is obtained as

\[
V_a = \int_a^\infty E \, dr = \int_a^\infty \frac{I}{2\pi \sigma r^2} \, dr = \frac{I}{2\pi \sigma a}.
\]

Is it possible to define the resistance of this grounding system? We need two terminals
for a resistor, and the hemispherical electrode has (seemingly) just one. However, the current
must be collected somewhere at a distant point by a generator and returned to the electrode
through a wire. The distant point is the other “resistor” contact. Usually this other contact is a
large grounding system of a power plant, with a large contact area with the ground, so that the
principal contribution to the resistance of this resistor comes from the hemispherical electrode.
We can therefore define the resistance of the hemispherical grounding electrode as

\[
R = \frac{V_a}{I} = \frac{1}{2\pi \sigma a}.
\]

The potential at any point on the surface of the earth due to the current flow is

\[
V_r = \int_r^\infty E(r) \, dr = \frac{I}{2\pi \sigma r},
\]

and the potential difference between two points at a distance \( \Delta r = d \) apart is given by

\[
\Delta V = \frac{I}{2\pi \sigma r} - \frac{I}{2\pi \sigma (r + d)}.
\]

Suppose a person approaches the grounding electrode when a large current of \( I = 1000 \, \text{A} \)
is flowing through it. The potential difference between his feet can be very large and
even fatal. For example, if \( \sigma = 10^{-2} \, \text{S/m} \), \( r = 1 \, \text{m} \), and the person’s step is \( d = 0.75 \, \text{m} \) long, the
potential difference between the two feet will be 6820 volts.
Real grounding electrodes are usually in the shape of a plate, a rod, or a thin metal mesh and are buried in the ground. The variation in the conductivity of the soil has a large effect on the behavior of the grounding electrode. For that reason the conductivity around it is sometimes purposely increased by, for example, adding salt to the soil. The example of a spherical electrode is useful because it gives us an idea of the order of magnitude, but it is certainly not precise.

Questions and problems: Q10.14 to Q10.17, P10.17 to P10.19

10.8 Chapter Summary

1. The basic quantities that describe electric current are the current density vector, \( \mathbf{J} \), describing current flow at any point, and the current intensity, \( I \), describing current flow through a surface.

2. The current density vector \( \mathbf{J} \) is most frequently a linear function of the local electric field strength, \( \mathbf{J} = \sigma \mathbf{E} \) (point form of Ohm’s law), also expressed as \( \mathbf{E} = \rho \mathbf{J} \), where \( \sigma \) is the conductor conductivity, and \( \rho \) its resistivity.

3. The volume power density of transformation of electric energy into heat in conductors is described by the point form of Joule’s law, \( p_j = \mathbf{J} \cdot \mathbf{E} \).

4. The current continuity equation is a mathematical expression of the experimental law of conservation of electric charges. Its form for time-invariant currents is just the Kirchhoff’s current law of circuit theory.

5. A resistor is an element, made of a resistive material, with two terminals, each equipotential. Ohm’s and Joule’s laws for resistors known from circuit theory are derived from field theory.

6. It is not possible to maintain a current field without devices known as electric generators, which convert some other form of energy into electric energy, i.e., energy of separated electric charges. Electric generators are characterized by their electromotive force and internal resistance.

7. Based on boundary conditions for current fields, an image method can be formulated for these fields, similar to that in electrostatics. However, in this case the boundary conditions are satisfied by a “positive” image. The image method can be used, for example, to determine the field and resistance of grounding electrodes.

Questions

Q10.1. What do you think is the main difference between the motion of a fluid and the motion of charges constituting an electric current in conductors?

Q10.2. Describe in your own words the mechanism of transformation of electric energy into heat in current-carrying conductors.

Q10.3. Is Eq. (10.5) valid also for a closed surface, or must the surface be open? Explain.
Q10.4. A closed surface $S$ is situated in the field of time-invariant currents. What is the charge that passes through $S$ during a time interval $dt$?

Q10.5. Is a current intensity on the order of 1 A frequent in engineering applications? Are current densities on the order of 1 mA/m$^2$ or 1 kA/m$^2$ frequent in engineering applications? Explain.

Q10.6. What is the difference between linear and nonlinear resistors? Can you think of an example of a nonlinear resistor?

Q10.7. A wire of length $l$, cross-sectional area $S$, and resistivity $\rho$ is made to meander very densely. The lengths of the successive parts of the meander are on the order of the wire radius. Is it possible to evaluate the resistance of such a wire accurately using Eq. (10.19)? Explain.

Q10.8. Explain in your own words the statement in Eq. (10.20).

Q10.9. Assume that you made a resistor in the form of an uninsulated metal container (one resistor contact) with a conducting liquid (e.g., tap water with a small amount of salt), and a thin wire dipped into the liquid (the other resistor contact). If you change the level of water, but keep the length of the wire in the liquid constant, will this produce a substantial variation of the resistor resistance? If you change the length of the wire in the liquid, will this produce a substantial variation of the resistor resistance? Explain.

Q10.10. What is meant by "nonelectric forces" acting on electric charges inside electric generators?

Q10.11. List a few types of electric generators, and explain the nonelectric (impressed) forces acting in them.

Q10.12. Does the impressed electric field strength describe an electric field? What is the unit of the impressed electric field strength?

Q10.13. Prove that on a boundary surface in a time-invariant current field, $\mathbf{J}_{1\text{norm}} = \mathbf{J}_{2\text{norm}}$.

Q10.14. Where do you think the charges producing the electric field in the ground in Fig. 10.9 are located? (These charges cause the current flow in the ground.)

Q10.15. Explain in your own words what is meant by the grounding resistance, and what this resistor is physically.

Q10.16. Is it possible to define the grounding resistance if the generator is not grounded? Explain.

Q10.17. Assume that a large current is flowing through the grounding electrode. Propose at least three different ways to approach the electrode with a minimum danger of electric shock.

**PROBLEMS**

P10.1. Prove that the current in any homogeneous cylindrical conductor is distributed uniformly over the conductor cross section.

P10.2. Uniformly distributed charged particles are placed in a liquid dielectric. The number of particles per unit volume is $N = 10^9$ m$^{-3}$, and each is charged with $Q = 10^{-16}$ C. Calculate the current density and the current magnitude obtained when such a liquid moves with a velocity of $v = 1.2$ m/s through a pipe of cross section area $S = 1$ cm$^2$. Is this current produced by an electric field?
P10.3. A conductive wire has the shape of a hollow cylinder with inner radius \( a \) and outer radius \( b \). A current \( I \) flows through the wire. Plot the current density as a function of radius, \( J(r) \). If the conductivity of the wire is \( \sigma \), what is the resistance of the wire per unit length?

P10.4. A conductor of radius \( a \) is connected to one with radius \( b \). If a current \( I \) is flowing through the conductor, find the ratio of the current densities and of the densities of Joule's losses in both parts of the conductor if the conductivity for both parts is \( \sigma \).

P10.5. The homogeneous dielectric inside a coaxial cable is not perfect. Therefore, there is some current, \( I = 50 \mu A \), flowing through the dielectric from the inner toward the outer conductor. Plot the current density inside the cable dielectric, if the inner conductor radius is \( a = 1 \text{ mm} \), the outer radius \( b = 7 \text{ mm} \), and the cable length \( l = 10 \text{ m} \).

P10.6. Find the expression for the current through the rectangular surface \( S \) in Fig. P10.6 as a function of the surface width \( x \).

![Figure P10.6 Calculating current intensity](image)

P10.7. Find the expression for the current intensity through a circular surface \( S \) shown in Fig. P10.7 for \( 0 < r < \infty \).

![Figure P10.7 A coaxial cable](image)
P10.8. The resistivity of a wire segment of length \( l \) and cross-sectional area \( S \) varies along its length as \( \rho(x) = \rho_0(1 + x/l) \). Determine the wire segment resistance.

P10.9. Find the resistance between points 2 and 2' of the resistor shown in Fig. P10.9.

![Figure P10.9 An idealized resistor cross section](image)

Figure P10.9 An idealized resistor cross section

![Figure P10.10 Two inhomogeneous conductors of radius \( a \)](image)

Figure P10.10 Two inhomogeneous conductors of radius \( a \).

P10.10. Show that the electric field is uniform in the case of both inhomogeneous conductors in Fig. P10.10. Find the resistance per unit length of these conductors, the ratio of currents in the two layers, and the ratio of Joule's losses in the two layers.

P10.11. The dielectric in a coaxial cable with inner radius \( a \) and outer radius \( b \) has a very large, but finite, resistivity \( \rho \). Find the conductance per unit length between the cable conductors. Specifically, find the conductance between the conductors of a cable \( L = 1 \) km long with \( a = 1 \) cm, \( b = 3 \) cm, and \( \rho = 10^{31} \Omega \cdot \text{m} \).

P10.12. A lead battery is shown schematically in Fig. P10.12. The total surface area of the lead plates is \( S = 3.2 \) dm\(^2\), and the distance between the plates is \( d = 5 \) mm. Find the approximate internal resistance of the battery, if the resistivity of the electrolyte is \( \rho = 0.016 \Omega \cdot \text{m} \).

![Figure P10.12 A lead battery](image)

Figure P10.12 A lead battery
P10.13. Calculate approximately the resistance between cross sections 1 and 2 of the nonuniform strip conductor sketched in Fig. P10.13. The resistivity of the conductor is \( \rho \). Why can the resistance be calculated only approximately?

![Figure P10.13 A nonuniform strip conductor](image)

P10.14. Calculate approximately the resistance between cross sections 1 and 2 of the conical part of the conductor sketched in Fig. P10.14. The resistivity of the conductor is \( \rho \).

![Figure P10.14 A conical conductor](image)

![Figure P10.15 A resistor with an impressed field](image)

P10.15. In the darker shaded region of the very large conducting slab of conductivity \( \sigma \) and permittivity \( \varepsilon_0 \) shown in Fig. P10.15, a uniform impressed field \( E_i \) acts as indicated. The end surfaces of the slab are coated with a conductor of conductivity much greater than \( \sigma \), and connected by a wire of negligible resistance. Determine the current density, the electric field intensity, and the charge density at all points of the system. Ignore the fringing effect.

P10.16. Repeat problem P10.15 assuming that the permittivity of the slab is \( \varepsilon \) (different from \( \varepsilon_0 \)). Note that in that case polarization charges are also present.
P10.17. Determine the resistance of a hemispherical grounding electrode of radius $a$ if the ground is not homogeneous, but has a conductivity $\sigma_1$ for $a < r < b$, and $\sigma_2$ for $r > b$, where $b > a$, and $r$ is the distance from the grounding electrode center.

P10.18. Determine the resistance between two hemispherical grounding electrodes of radii $R_1$ and $R_2$, which are a distance $d$ ($d \gg R_1, R_2$) apart. The ground conductivity is $\sigma$.

P10.19. A grounding sphere of radius $a$ is buried at a depth $d$ ($d \gg a$) below the surface of the ground of conductivity $\sigma$ (Fig. P10.19). Determine the points at the surface of the ground at which the electric field intensity is the largest. Determine the electric field intensity at these points if the intensity of the current through the grounding sphere is $I$.

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Figure P10.19 A deeply buried spherical electrode