Chapter 4 Discrete-Time System Representation

Before any controller can be designed, one needs to obtain a model of the physical process that is to be controlled. The techniques to represent a continuous-time process in discrete-time framework are the focus of this chapter.

We will first extend the pulse transfer function concept form the previous chapter to deriving transfer functions for systems with mixed discrete-time and continuous-time elements. We will be using the starred Laplace transform introduced in Chapter 2 to bridge the discrete and continuous domain. The discrete-time representation of a continuous-time plant transfer function preceded with a zero-order-hold will be used extensively throughout the course. Finally, we will introduce the discrete representation of plant model described in state-space form. A continuous-time state-space model can be represented as a set of first-order difference equations at the sample instants. The relationship between the transfer function representation and the state-space representation will be explored.

4.1 Pulse Transfer Function

In analyzing computer controlled systems, we often encounters signals in the system that are impulse sampled and others are not. To obtain pulse transfer functions and to analyze discrete-time control systems, we must be able to obtain the transform of the output signals of systems that contain sampling operations in various places in the loops.

Before we start, let's revisit the concept of impulse sampling described in Chapter 2. Recall that the sample operation, Figure 4.1, can be considered as convolution between a continuous-time signal $x(t)$ with the pulse train $\delta_T(t)$ (see Eq. (2.1))

$$x^*(t) = \sum_{k=-\infty}^{\infty} x(t)\delta(t-kT) = \sum_{k=-\infty}^{\infty} \delta(t-kT)\cdot x(kT) \qquad (4.1)$$

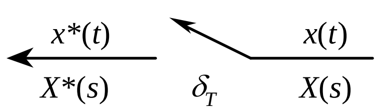


Figure 4.1 Impulse Sampling

The Laplace transform of the sampled signal $x^*(t)$ is (see Eq. (2.2))

$$X^*(s) = \mathcal{L}\left[x^*(t)\right] = \int_0^{\infty} x^*(\tau)\cdot e^{-s\tau}d\tau = \sum_{k=0}^{\infty} x(kT)\cdot e^{-kTs} \qquad (4.2)$$

Note that we are using the one-sided Laplace transform. We will be referring the Laplace transform of a sampled signal by the *starred Laplace transform*. Compare Eq. (4.2) with the definition of the z-transform, we see that the z-transform can be understood as the starred Laplace transform with e^{Ts} replaced by z. The z-transform can also be viewed as a shorthand notation for the starred Laplace transform.

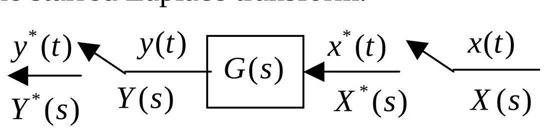


Figure 4.2 Impulse Sampled System

Suppose the impulse sampler is followed by a linear continuous-time element whose transfer function is $G(s)$, as shown in Figure 4.2. In the following analysis, we will assume that all the initial conditions are zero in the system. Notice that we cannot find a transfer function from $X(s)$ to $Y(s)$. However, it is true that the output of the system Y(s) is

$$Y(s) = G(s) \cdot X^*(s) \qquad (4.3)$$

In the following derivation, we shall show that in taking the starred Laplace transform of Eq. (4.3) we may factor out $X^*(s)$ so that

$$Y^*(s) = \left[G(s) \cdot X^*(s)\right]^* = \left[G(s)\right]^* \cdot X^*(s) = G^*(s) \cdot X^*(s) \quad (4.4)$$

This fact is very important in deriving the pulse transfer function for computer-controlled systems. To derive Eq. (4.4), note that

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[G(s) \cdot X^*(s)\right] = \int_0^t g(t-\tau) \cdot x^*(\tau)\, d\tau \\ &= \int_0^t g(t-\tau) \cdot \sum_{k=0}^{\infty} x(\tau)\delta(\tau - kT)\, d\tau = \sum_{k=0}^{\infty} \int_0^t g(t-\tau)x(\tau) \cdot \delta(\tau - kT)\, d\tau \\ &= \sum_{k=0}^{\infty} g(t-kT) \cdot x(kT) \end{aligned}$$

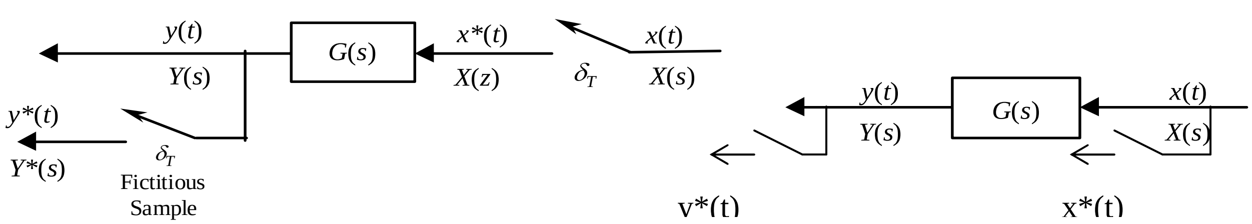
Take z-transform of $y(t)$, we get

$$\begin{aligned} Y(z) &= Z\left[y(t)\right] = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty} g(nT-kT)x(kT)\right] \cdot z^{-n} = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} g(nT-kT)x(kT)\right] \cdot z^{-n} \\ &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} g(nT-kT) \cdot x(kT) \cdot z^{-n} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} g(mT) \cdot x(kT) \cdot z^{-(k+m)}, \qquad \text{where } m = n-k \\ &= \left[\sum_{m=0}^{\infty} g(mT) \cdot z^{-m}\right] \cdot \left[\sum_{k=0}^{\infty} x(kT) \cdot z^{-k}\right] = G(z) \cdot X(z) \end{aligned}$$

Since the z-transform can also be viewed as a shorthand notation for the starred Laplace transform, the above equation can also be expressed as

$$Y^*(s) = G^*(s) \cdot X^*(s)$$

To summarize, Eq. (4.4) states that in taking the starred Laplace transform of a product of two transforms, where one is an ordinary Laplace transform and the other is a starred Laplace transform, the starred transform can be factored out of the starred Laplace transform operation.



(b)

Figure 4.3 (a) Continuous-Time System with Input Sampler; (b) Continuous-Time System

The pulse transfer function $G(z)$ of the system shown in Figure 4.3(a) is

$$G(z) = \frac{Y(z)}{X(z)} = \mathcal{Z}\left[G(s)\right] = \mathcal{Z}\left[\mathcal{L}^{-1}\left[G(s)\right]\Big|_{t=kT}\right] \tag{4.5}$$

However, for the system shown in Figure 4.3 (b), the Laplace transform of the output $y(t)$ is

$$Y(s) = G(s)\cdot X(s)$$

The corresponding starred Laplace transform yields

$$Y^*(s) = \left[G(s)X(s)\right]^* = \left[GX(s)\right]^*$$

or, in terms of the z-transform

$$Y(z) = \mathcal{Z}\left[Y(s)\right] = \mathcal{Z}\left[G(s)X(s)\right] = \mathcal{Z}\left[GX(s)\right] = GX(z) \neq G(z)X(z)$$

The fact that the z-transform of $G(s)X(s)$ is not equal to $G(z)X(z)$ results from the fact that the inputs to the system $G(s)$ are different. $X(s)$ and $X^*(s)$ are two different signals with very different frequency characteristics. The output $Y(s)$ under the two inputs thus are different.

In discussing the pulse transfer function, we assume that the element in consideration is preceded by an input sampler. The presence of an output sampler does not affect the pulse transfer function. If an output sampler is not physically present, it is always possible to assume that a fictitious sampler is present at the output. This means that although the output signal is continuous, we can consider the values of the output only at the sample instants and get the output sequence $y(kT)$.

In summary, if the input to a continuous-time system $G(s)$ is an impulse-sampled signal, then the corresponding pulse transfer function of G(s) is given by

$$G(z) = Z\left[G(s)\right] = Z\left[\mathbf{L}^{-1}\left[G(s)\right]\Big|_{t=kT}\right]. \tag{4.6}$$

Example 4.1 *Calculating Pulse Transfer Function (Continuous-time system with sampler)*
Obtain the pulse transfer function $G(z)$ of the system shown in Figure 4.3 (a), where $G(s)$ is given by

$$G(s) = \frac{1}{s+a}$$

Solution:

Since there is an input sampler at the input end of $G(s)$, the pulse transfer function is $G(z) = \mathcal{Z}\left[G(s)\right]$. There are two methods to calculate $\mathcal{Z}\left[G(s)\right]$:

<u>*Method I*</u>: (Table lookup)

From the z-transform table (Table 3.1), we have

$$G(z) = \mathcal{Z}\left[\frac{1}{s+a}\right] = \frac{z}{z - e^{-aT}} = \frac{1}{1 - e^{-aT} z^{-1}}$$

__Method II__: (Direct Calculation)

The impulse response for $G(s)$ is

$$g(t) = \mathcal{L}^{-1}\left[G(s)\right] = e^{-at}$$

Hence

$$g(kT) = e^{-akT}, \quad k = 0, 1, 2, \ldots$$

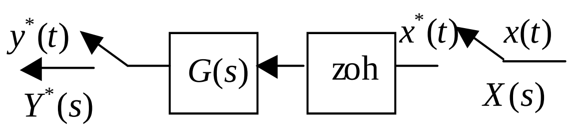
Therefore

$$G(z) = \mathcal{Z}\left[g(kT)\right] = \sum_{k=0}^{\infty} e^{-akT} \cdot z^{-k} = \sum_{k=0}^{\infty} \left(e^{aT} z\right)^{-k} = \frac{1}{1 - \left(e^{aT} z\right)^{-1}} = \frac{1}{1 - e^{-aT} z^{-1}}$$

Example 4.2 *Calculating Pulse Transfer Function*

Obtain the pulse transfer function $G(z)$ of the system shown below, where $G(s)$ is given by

$$G(s) = \frac{1}{s(s+1)}$$



Solution:

Since there is an input sampler at the input of $G(s)$, the pulse transfer function is $G(z) = \mathcal{Z}\left[G(s)\right]$.

$$G(z) = \mathcal{Z}\left[G(s)\right] = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s} \frac{1}{s(s+1)}\right] = \mathcal{Z}\left[\left(1 - e^{-Ts}\right) \frac{1}{s^2(s+1)}\right]$$

$$= \left(1 - z^{-1}\right) \mathcal{Z}\left[\frac{1}{s^2(s+1)}\right] = \left(1 - z^{-1}\right) \mathcal{Z}\left[\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right]$$

From Table 3.1, the z-transform of each of the partial fraction expansion terms can be obtained. Thus

$$G(z) = \left(1 - z^{-1}\right)\left[\frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z - e^{-T}}\right] = \frac{(T - 1 + e^{-T})z + (1 - e^{-T} - Te^{-T})}{(z-1)(z - e^{-T})}$$

Closed-Loop Transfer Function of Computer Controlled Systems

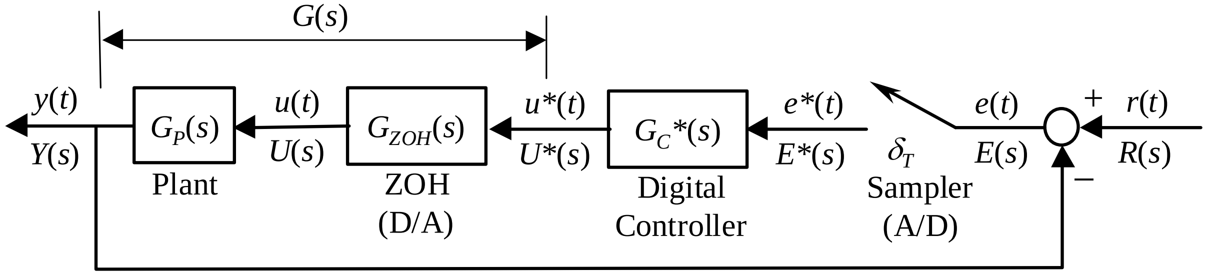


Figure 4.4 Block Diagram of a Computer Controlled System

Figure 4.4 shows the block diagram of a typical computer controlled system. The sampler, digital controller, and zero-order hold (ZOH) together produce a continuous-time (piecewise-constant) control signal $u(t)$ to be fed to the continuous-time plant $G_P(s)$. The transfer function of the digital controller is shown as $G_C*(s)$. In actual implementation, the controller solves a linear constant coefficient difference equation whose input/output relationship is given by the pulse transfer function $G_C(z)$.

Again, notice that we cannot find a transfer function from *R(s)* to *Y(s)*. However, we can find that for the signals from *R*(s)* to *Y*(s)*. Before we start deriving the closed-loop pulse transfer function, lets us define an augmented plant transfer function

$$G(s) = G_P(s) \cdot G_{ZOH}(s) = G_P(s) \cdot \frac{1 - e^{-Ts}}{s} \qquad (4.7)$$

From Figure 4.4, we see that

$$Y(s) = G(s) \cdot G_C*(s) \cdot E*(s) \quad \text{or} \quad Y*(s) = G*(s) \cdot G_C*(s) \cdot E*(s)$$

Using the z-transform notation, we get

$$Y(z) = G(z) \cdot G_C(z) \cdot E(z)$$

where $G(z) = \mathcal{Z}\left[G(s)\right]$. Since $E(z) = R(z) - Y(z)$, we have

$$Y(z) = G(z) \cdot G_C(z) \cdot \left[R(z) - Y(z)\right].$$

Therefore, the closed-loop pulse transfer function of the digital control system shown in Figure 4.4 is given by

$$\frac{Y(z)}{R(z)} = \frac{G(z) \cdot G_C(z)}{1 + G(z) \cdot G_C(z)} \qquad (4.8)$$

The overall performance of the closed-loop system can be manipulated by the proper selection of the digital controller $G_C(z)$.

Note that Eq. (4.**Error! Reference source not found.**) is also the closed-loop transfer function of the discrete-time system shown in Figure 4.5. The responses of the two systems shown in Figure 4.4 and in Figure 4.5 will have the same values at the sample instants if the discretized plant $G(z)$ is properly selected.

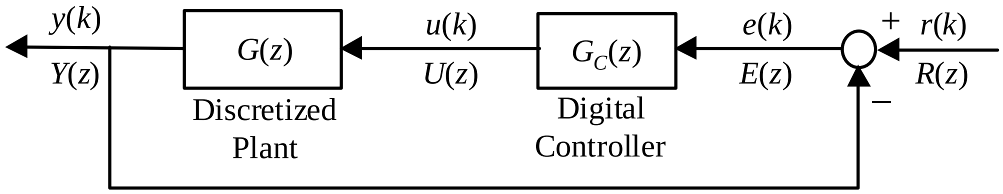


Figure 4.5 Discrete-time Closed-Loop Control System

4.2 Discrete Input/Output Representation of Sampled-Data Systems

In most control systems, the plant to be controlled by microprocessor is one that is described by differential equations. To analyze the digital control of such systems, a discrete-time description of the plant to be controlled must first be obtained.

A continuous-time single-input single-output (SISO) plant represented by the transfer function $G_P(s)$ is shown in Figure 4.6. Under digital control, the plant is preceded by a zero-order hold device to convert the discrete output of the digital controller to continuous-time (piecewise-step) signal.

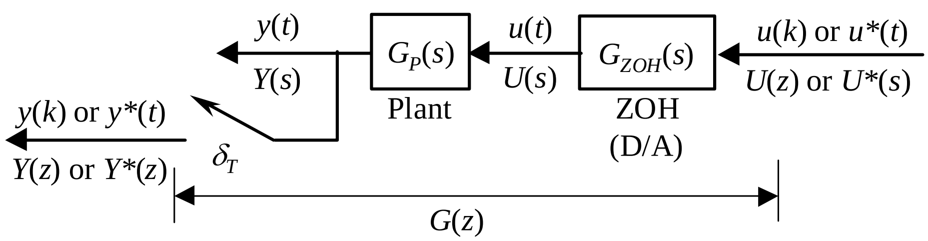


Figure 4.6 Continuous-Time Plant to be Digitally Controlled

The input/output pulse transfer function of the plant seeing by the digital controller can be derived using the starred Laplace transform process described in the previous section.

From Figure 4.6, note that

$$Y(s) = G_P(s)\cdot G_{ZOH}(s)\cdot U^*(s)$$

Taking the starred Laplace transform of the above equation we get

$$Y^*(s) = \left[G_P(s)\cdot G_{ZOH}(s)\right]^*\cdot U^*(s)$$

In the z-transform notation,

$$Y(z) = \mathcal{Z}\left[G_P(s)\cdot G_{ZOH}(s)\right]\cdot U(z)$$

Hence, the discrete-time plant model is

$$\frac{Y(z)}{U(z)} = \mathcal{Z}\left[G_P(s)\cdot G_{ZOH}(s)\right] = G(z) \qquad (4.9)$$

To calculate $G(z)$, first recall that the transfer function for the zero-order hold is

$$G_{ZOH}(s) = \frac{1-e^{-Ts}}{s}$$

Substitute $G_{ZOH}(s)$ into Eq. (4.**Error! Reference source not found.**) we have

$$G(z) = \mathrm{Z}\left[\frac{1-e^{-Ts}}{s}G_P(s)\right] = \mathrm{Z}\left[\frac{G_P(s)}{s}\right] - \mathrm{Z}\left[e^{-Ts}\frac{G_P(s)}{s}\right] = \mathrm{Z}\left[\frac{G_P(s)}{s}\right] - z^{-1}\cdot\mathrm{Z}\left[\frac{G_P(s)}{s}\right]$$

Hence the pulse transfer function of the plant is

$$G(z) = (1-z^{-1})\mathrm{Z}\left[\frac{G_P(s)}{s}\right] = \frac{z-1}{z}\mathrm{Z}\left[\frac{G_P(s)}{s}\right] \quad (4.10)$$

This relation will be used extensively in deriving the discrete-time plant model for direct digital control.

Example 4.3 *ZOH Equivalent Discrete Transfer Function*
Find the ZOH equivalent discrete transfer function of the following continuous-time plants:

1. $G_P(s) = \frac{K}{s+a}$ 2. $G_P(s) = \frac{1}{s^2}$

Solution:

Using Eq. (4.**Error! Reference source not found.**) we see that

$$G(z) = \left(1-z^{-1}\right)\mathrm{Z}\left[\frac{K}{s(s+a)}\right] = \left(1-z^{-1}\right)\mathrm{Z}\left[\frac{K}{a}\left(\frac{1}{s}-\frac{1}{s+a}\right)\right] = \frac{z-1}{z}\frac{K}{a}\left(\frac{z}{z-1}-\frac{z}{z-e^{-aT}}\right)$$

The above can be further simplified to

$$G(z) = \frac{K}{a}\frac{\left(1-e^{-aT}\right)}{z-e^{-aT}}$$

Using Eq. (4.**Error! Reference source not found.**) we have

$$G(z) = \left(1-z^{-1}\right)\mathrm{Z}\left[\frac{1}{s^3}\right] = \frac{z-1}{z}\frac{T^2}{2}\frac{z(z+1)}{(z-1)^3} = \frac{T^2}{2}\frac{z+1}{(z-1)^2}$$

The MATLAB function `c2d.m` computes Eq. (4.**Error! Reference source not found.**) (the ZOH equivalent is the default) as well as other discretization methods that we will be discussing the later chapters.

System with Time Delay (Transport Lag)

Continuous-time systems with time delay are common especially for chemical process-control plants. The time delay is often attributed to the finite time needed to transport fluids or materials between processes and the controls and/or sensors. Computation delay is an inherent characteristics of computer control and has the same effect as if the process has a pure (sub-sampling interval) delay. Although the pulse transfer function only deal with the input/output relationship at sample instants, with the technique shown here, it is possible to obtain the discrete transfer function of such processes.

Consider a continuous-time plant with time delay of T_D seconds whose continuous-time transfer function is expressed as

$$G_P(s) = e^{-T_D s} \cdot G(s)$$

The time delay T_D includes both the process time delay and the computation delay of the controller. To obtain the ZOH equivalent discrete-time transfer function of the system, we need to define a positive integer n and a positive number T_L less than 1, such that

$$T_D = nT + T_L .$$

Substitute $G_P(s)$ into Eq. (4.**Error! Reference source not found.**) and employ the definition above, we get

$$G(z) = \left(1 - z^{-1}\right) Z\left[\frac{G_P(s)}{s}\right] = \left(1 - z^{-1}\right) Z\left[e^{-nTs}\, \frac{e^{-T_L s} \cdot G(s)}{s}\right]$$

Since n is an integer, the term e^{-nT} reduces to z^{-n} when taking the z-transform. Hence,

$$G(z) = \left(1 - z^{-1}\right) z^{-n} \mathrm{Z}\left[\frac{e^{-T_L s} \cdot G(s)}{s}\right] = \frac{z-1}{z^{n+1}} \mathrm{Z}\left[\mathbf{L}^{-1}\left[\frac{e^{-T_L s} \cdot G(s)}{s}\right]\Bigg|_{t=kT}\right] \qquad (4.11)$$

Since $T_L < T$, the z-transform of Eq.(4.11) can be calculated directly by using inverse Laplace transformation in conjunction with the z-transform.

Example 4.4 *ZOH Equivalent with Time Delay*

Consider the following first order process control model with a 0.5 sec time delay

$$G_P(s) = \frac{e^{-0.5s}}{s+1}$$

Find the equivalent ZOH transfer function with a sampling time of 0.4 sec.

Solution:

Since T_D = 0.5 sec, we can select n = 1 and T_L = 0.1. Substitute the continuous plant transfer function into Eq. (4.**Error! Reference source not found.**) we obtain

$$G(z) = \frac{z-1}{z^{1+1}} \mathrm{Z}\left[\frac{e^{-0.1s}}{s(s+1)}\right] = \frac{z-1}{z^2} \mathrm{Z}\left[e^{-0.1s}\left(\frac{1}{s} - \frac{1}{s+1}\right)\right]$$

To calculate the z-transform, we need to find the inverse Laplace transformation of the two signals in the bracket. The first term is a unit step sequence delayed by 0.1sec, hence

$$Z\left[e^{-0.1s}\,\frac{1}{s}\right] = \frac{1}{z-1}$$

The second term is a delayed exponential signal of the form $e^{-(kT-0.1)}$, hence the corresponding z-transform is

$$Z\left[e^{-0.1s}\frac{1}{s+1}\right]=Z\left[U_1(t-0.1)e^{-(t-0.1)}\right]=\frac{1}{z}Z\left[e^{-kT-0.4+0.1}\right]=\frac{1}{z}Z\left[e^{-0.4k-0.4+0.1}\right]=\frac{e^{-0.3}}{z-e^{-0.4}}$$

Combine the two terms, we get

$$G(z)=\frac{z-1}{z^2}\mathrm{Z}\left[\frac{e^{-0.1s}}{s(s+1)}\right]=\frac{z-1}{z^2}\left(\frac{1}{z-1}-\frac{e^{-0.3}}{z-e^{-0.4}}\right)$$

Further simplification we get

$$G(z)=\frac{z-1}{z^2}\left(\frac{1}{z-1}-\frac{e^{-0.3}}{z-e^{-0.4}}\right)=\frac{0.2592z+0.0705}{z^2\left(z-0.6703\right)}.$$

We can also use the Table of the modified Z-Transforms to find the following term:

$$\mathrm{Z}\left[\frac{e^{-0.1s}}{s(s+1)}\right]=F(z,m),\text{ where } F(s) = \frac{1}{s(s+1)},\text{ and } m\triangleq 1-\frac{T_d}{T}=1-\frac{0.1}{0.4}=0.75.$$

From the Table, we identify the pair that

$$F(s) = \frac{1}{s(s+a)},\quad F(z,m)=\frac{(1-e^{-amT})z+(e^{-amT}-e^{-aT})}{(z-1)((z-e^{-aT})}.$$

By plugging in T=0.4, a=1, and m=0.75, we obtain the same result.

Alternatively, the following Matlab commands would generate the same result.

```
G=tf([1],[1 1])
T=0.4 %Sampling time: 0.4
set(G,'InputDelay',0.5)  % add delay to the system

G

%Transfer function:
%                 1
%exp(-0.5*s) * -----
%               s + 1

Gd=c2d(G,T,'zoh')


%Transfer function:
%          0.2592 z + 0.0705
%z^(-2) * -----------------
%             z - 0.6703
```

4.3 Inter-Sample Ripples

The modeling of the a computer controlled system in Figure 4.4 yields a discrete-time system model in Figure 4.5, where the exact response at the sampling points can be derived. However this discrete-time model does not tell what happen between the sampling points. In some cases, the system may exhibit inter-sample ripples that are not apparent at the sampling points. In this section we derive the model to describe the inter-sample response.

Consider a continuous time signal f(t) and it's sampled signal f*(t) or f(kT). Suppose we want to represent inter-sample singal f(kT-T_d), where $0 < T_d < T$. This signal would be the sampling of f(t-T_d), i.e. f*(t- T_d). Note that f*(t- T_d) is a discrete-time signal, where a time shift of T_d allows peeking the signal at time kT- T_d. Now consider the digital cotrol system in Figuure 4.4, the fractional delay can be added to any signal of interest. Particularly, we add the delay at the output y(t) and obtain y(t- T_d), denoted as y_d(t) and then sample the signal to get y_d*. To obtain y_d*(t) as it relates to the input r(t), Note again that

$$Y(s) = G_P(s)\cdot G_{ZOH}(s)\cdot U^*(s) = G(s)\cdot \frac{C^*(s)}{1+G^*(s)C^*(s)}R^*(s)$$

The Laplace transform of y*(t-D) is

$$Y_d^*(s) = \left(G(s)e^{-sT_d}\cdot \frac{C^*(s)}{1+G^*(s)C^*(s)}R^*(s)\right)^* = \left(G(s)e^{-sT_d}\right)^*\cdot \frac{C^*(s)}{1+G^*(s)C^*(s)}R^*(s) \qquad (4.12)$$

Thus we obtain the pulsed transfer function

$$\frac{Y_d^*(s)}{R^*(s)} = \frac{\left(G(s)e^{-sT_d}\right)^* C^*(s)}{1+G^*(s)C^*(s)} \qquad (4.13)$$

Or equivalently

$$\frac{Y_D(z)}{R(z)} = \frac{G_D(z)C(z)}{1+G(z)C(z)}, \quad G_D(s) = G(s)e^{-ST_D} \qquad (4.14)$$

Or, by the modified Z transform:

$$\frac{Y_d(z)}{R(z)} = \frac{G(z,m)C(z)}{1+G(z)C(z)}, \quad m = 1-\frac{T_d}{T} \qquad (4.15)$$

We have obtained the relation between the sampled reference input and the sampled inter-sample output. While the delay T_d may be any positive value less than the sampling time, it usually suffice to select a finite numbers evenly spaced between the sampling instants, e.g.

$$T_d = \frac{1}{nT}, \frac{2}{nT}, \ldots, \frac{n-1}{nT}.$$

Example 4.5 *Inter-Sample Ripples*

Consider the following digital control system:

$$G_P(s) = \frac{2500}{s^2 + 10s + 2500}$$

$$C(z) = \frac{4.353\text{ z - }3.49}{\text{z + }0.9608}$$

with sampling time T=0.01 second.

The discrete transfer fuction Y(z)/R(z) has been calculated and its impulse response shown in Figure 4.7 dipicts non-oscillatory behavior. The discrete transfer functions for inter-sample points have also been calculated by Equation (4.14) for $T_d = \frac{1}{4T}, \frac{2}{4T}, \ldots, \frac{3}{4T}$, respectively. Figure 4.7 clearly shows inter-sample ripples (hidden oscillation) that are hidden from the at-sample response.

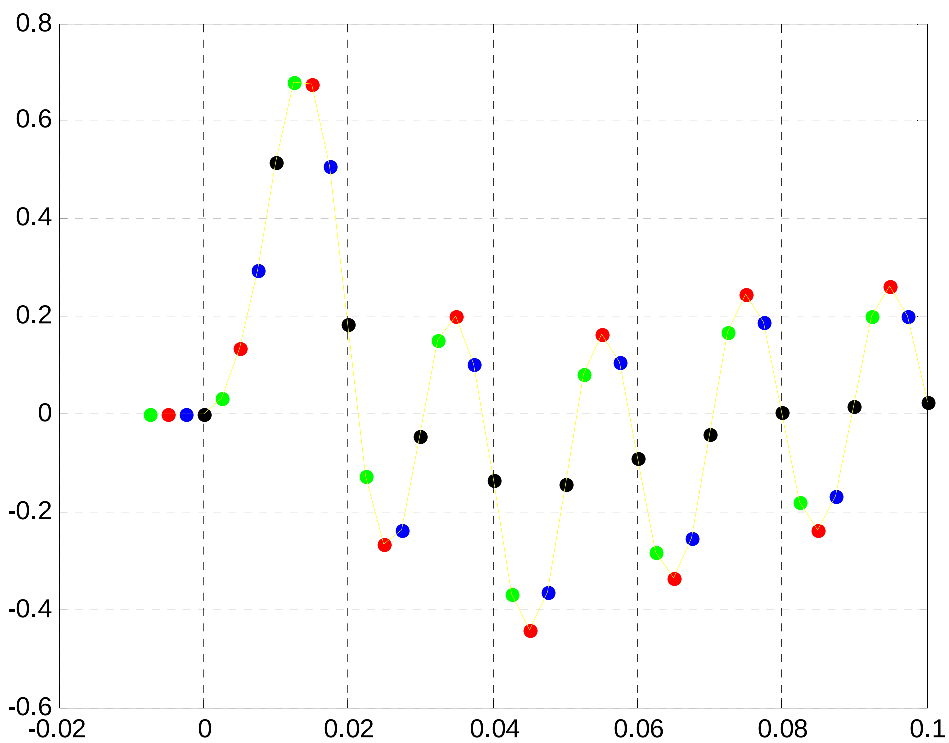


Figure 4.7 Impulse response of the closed loop computer controlled system Example 4.5. The black dots mak the at-sample response. Ohers dots mark the inter-sample response.

APPENDIX A. TABLE OF LAPLACE TRANSFORMS, z-TRANSFORMS AND MODIFIED z-TRANSFORMS

Laplace Transform $F(s)$	Time Function $f(t)$ $t > 0$	z-Transform $F(z)$	Modified z-Transform $F(z,m)$
1	$\delta(t)$	1	0
e^{-kTs}	$\delta(t-kT)$	z^{-k}	z^{-k-1+m}
$\frac{1}{s}$	$u_s(t)$	$\frac{z}{z-1}$	$\frac{1}{z-1}$
$\frac{1}{s^2}$	t	$\frac{Tz}{(z-1)^2}$	$\frac{mT}{z-1} + \frac{T}{(z-1)^2}$
$\frac{2}{s^3}$	t^2	$\frac{T^2 z(z+1)}{(z-1)^3}$	$T^2 \frac{m^2 z^2 + (2m - 2m^2 + 1)z + (m-1)^2}{(z-1)^3}$
$\frac{(k-1)!}{s^k}$	t^{k-1}	$\lim_{a \to 0} (-1)^{k-1} \frac{\partial^{k-1}}{\partial a^{k-1}} \left[\frac{z}{z - e^{-aT}}\right]$	$\lim_{a \to 0} (-1)^{k-1} \frac{\partial^{k-1}}{\partial a^{k-1}} \left[\frac{e^{-amT}}{z - e^{-aT}}\right]$
$\frac{1}{s+a}$	e^{-at}	$\frac{z}{z - e^{-aT}}$	$\frac{e^{-amT}}{z - e^{-aT}}$
$\frac{1}{(s+a)^2}$	te^{-at}	$\frac{Tze^{-aT}}{(z - e^{-aT})^2}$	$\frac{Te^{-amT}[e^{-aT} + m(z - e^{-aT})]}{(z - e^{-aT})^2}$
$\frac{(k-1)!}{(s+a)^k}$	$t^k e^{-at}$	$(-1)^k \frac{\partial^k}{\partial a^k} \frac{z}{z - e^{-aT}}$	$(-1)^k \frac{\partial^k}{\partial a^k} \left[\frac{e^{-amT}}{z - e^{-aT}}\right]$
$\frac{a}{s(s+a)}$	$1 - e^{-at}$	$\frac{z(1 - e^{-aT})}{(z-1)(z - e^{-aT})}$	$\frac{(1 - e^{-amT})z + (e^{-amT} - e^{-aT})}{(z-1)(z - e^{-aT})}$

Laplace Transform F(s)	Time Function f(t) t > 0	z-Transform F(z)	Modified z-Transform F(z,m)
$\frac{1}{(s+a)(s+b)}$	$\frac{1}{(b-a)}(e^{-at}-e^{-bt})$	$\frac{1}{(b-a)}\left[\frac{z}{z-e^{-aT}}-\frac{z}{z-e^{-bT}}\right]$	$\frac{1}{(b-a)}\left[\frac{e^{-amT}}{z-e^{-aT}}-\frac{e^{-bmT}}{z-e^{-bT}}\right]$
$\frac{a}{s^2(s+a)}$	$t-\frac{1}{a}(1-e^{-at})$	$\frac{Tz}{(z-1)^2}-\frac{(1-e^{-aT})z}{a(z-1)(z-e^{-aT})}$	$\frac{T}{(z-1)^2}+\frac{amT-1}{a(z-1)}+\frac{e^{-amT}}{a(z-e^{-aT})}$
$\frac{1}{(s+a)^2}$	te^{-at}	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$	$\frac{Te^{-amT}[e^{-aT}+m(z-e^{-aT})]}{(z-e^{-aT})^2}$
$\frac{a}{s^3(s+a)}$	$\frac{1}{2}(t^2-\frac{2}{a}t+\frac{2}{a^2}u_s(t)-\frac{2}{a^2}e^{-at})$	$\frac{T^2z}{(z-1)^3}+\frac{(aT-2)Tz}{2a(z-1)^2}+\frac{z}{a^2(z-1)}-\frac{z}{a^2(z-e^{-aT})}$	$\frac{T^2}{(z-1)^3}+\frac{T^2(m+½)a-T}{a(z-1)^2}+\frac{(amT)^2/2-amT+1}{a^2(z-1)}-\frac{e^{-amT}}{a^2(z-e^{-aT})}$
$\frac{a^2}{s(s+a)^2}$	$u_s(t)-(1+at)e^{-at}$	$\frac{z}{z-1}-\frac{z}{z-e^{-aT}}-\frac{aTe^{-aT}z}{(z-e^{-aT})^2}$	$\frac{1}{z-1}-\left[\frac{1+amT}{z-e^{-aT}}+\frac{aTe^{-aT}}{(z-e^{-aT})^2}\right]e^{-amT}$
$\frac{a^2}{s^2(s+a)^2}$	$t-\frac{2}{a}u_s(t)+(t+\frac{2}{a})e^{-at}$	$\frac{1}{a}\left[\frac{(aT+2)z-2z^2}{(z-1)^2}+\frac{2z}{z-e^{-aT}}+\frac{aTe^{-aT}z}{(z-e^{-aT})^2}\right]$	$\frac{1}{a}\left\{\frac{aT}{(z-1)^2}+\frac{amT-2}{z-1}+\left[\frac{aTe^{-aT}}{(z-e^{-aT})^2}-\frac{amT-2}{z-e^{-aT}}\right]e^{-amT}\right\}$
$\frac{\omega}{s^2+\omega^2}$	$\sin\omega t$	$\frac{z\sin\omega T}{z^2-2z\cos\omega T+1}$	$\frac{\sin m\omega T+\sin(1-m)\omega T}{z^2-2z\cos\omega T+1}$

Laplace Transform F(s)	Time Function f(t) t > 0	z-Transform F(z)	Modified z-Transform F(z,m)
$\frac{s}{s^2+\omega^2}$	$\cos\omega t$	$\frac{z(z-\cos\omega T)}{z^2-2z\cos\omega T+1}$	$\frac{\cos m\omega T-\cos(1-m)\omega T}{z^2-2z\cos\omega T+1}$
$\frac{\omega}{s^2-\omega^2}$	$\sinh\omega t$	$\frac{z\sinh\omega T}{z^2-2z\cosh\omega T+1}$	$\frac{\sinh m\omega T+\sinh(1-m)\omega T}{z^2-2z\cosh\omega T+1}$
$\frac{s}{s^2-\omega^2}$	$\cosh\omega t$	$\frac{z(z-\cosh\omega T)}{z^2-2z\cosh\omega T+1}$	$\frac{\cosh m\omega Tz-\cosh(1-m)\omega T}{z^2-2z\cosh\omega T+1}$
$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin\omega t$	$\frac{ze^{-aT}\sin\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$	$\frac{e^{-amT}[z\sin m\omega T+e^{-aT}\sin(1-m)\omega T]}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$
$\frac{a^2+\omega^2}{s[(s+a)^2+\omega^2]}$	$1-e^{-at}\sec\phi\cos(\omega t+\phi)$ $\phi=\tan^{-1}(-a/\omega)$	$\frac{z}{z-1}-\frac{z^2-ze^{-aT}\sec\phi\cos(\omega T-\phi)}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$	$\frac{1}{z-1}-\frac{e^{-maT}\sec\phi(Az-B)}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$ $A=\cos(m\omega T+\phi)$ $B=e^{-aT}\cos[(1-m)\omega T-\phi]$
$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos\omega t$	$\frac{z^2-ze^{-aT}\cos\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$	$\frac{e^{-maT}[z\cos m\omega T+e^{-aT}\sin(1-m)\omega T]}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$

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