Appendix 17: Delta Function

A17.1. General

A delta function (also called Dirac delta function) is a mathematical function, which is defined as:

\[ \delta(x) = \infty, \text{ for } x = 0 \]
\[ \delta(x) = 0, \text{ for } x \neq 0 \]

(A17.1.1)

\[ \text{and } \int_{-\infty}^{\infty} \delta(x) dx = 1 \]

i.e. a function which is only non-zero at \( x = 0 \) with a total area equal to one.

The delta function is symmetric:

\[ \delta(-x) = \delta(x) \]

(A17.1.2)

and when multiplied with a function \( F(x) \) and integrated, yields this function’s value at \( x = 0 \):

\[ \int_{-\infty}^{\infty} F(x) \delta(x) dx = F(0) \]

(A17.1.3)

Similarly, a shifted delta function will result in the function’s value at that point:

\[ \int_{-\infty}^{\infty} F(x) \delta(x - \xi) dx = F(\xi) \]

(A17.1.4)

A17.2. Solving Schrödinger’s equation with a delta function potential

The one-dimensional Schrödinger’s equation with a delta function potential with area, \( M \), and located at \( x = x_0 \) is as follows:

\[ -\frac{\hbar^2}{2m} \frac{d^2 \Psi}{dx^2} + M \delta(x - x_0) \Psi(x) = E \Psi(x) \]

(A17.2.1)
The general solutions are the same as for \( V(x) = 0 \) on either side of \( x = x_0 \):

\[
\Psi(x) = A \sin kx + B \cos kx \quad \text{for} \ x < x_0
\]

\[
\Psi(x) = C \sin kx + D \cos kx \quad \text{for} \ x > x_0
\]

with \( k = \frac{\sqrt{2mE}}{\hbar} \)  \hspace{1cm} (A17.2.2)

and the constants \( A, B, C \) and \( D \) must be determined from the boundary conditions.

The boundary condition at the delta function is obtained by integrating Schrödinger’s equation just around the delta function, yielding:

\[
\int_{x_0 - \varepsilon}^{x_0 + \varepsilon} \left( -\frac{\hbar^2}{2m} \frac{d^2\Psi}{dx^2} + M \delta(x - x_0) \Psi(x) dx = \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} E \Psi(x) dx \right) \hspace{1cm} (A17.2.3)
\]

Which reduces in the limit where \( \varepsilon \to 0 \) to:

\[
\left. \frac{d\Psi}{dx} \right|_{x_0 + \varepsilon} - \left. \frac{d\Psi}{dx} \right|_{x_0 - \varepsilon} = \frac{2m}{\hbar^2} M \Psi(x_0) \hspace{1cm} (A17.2.4)
\]

The boundary conditions at \( x = x_0 \) are: 1) the continuity of the wave function at \( x_0 \) and 2) a discontinuity of the derivative of the wave function at \( x_0 \) with the difference in slope given by A17.2.4.\(^1\)

The resulting equations are:

\[
\Psi(x_0) = A \sin kx_0 + B \cos kx_0 = C \sin kx_0 + D \cos kx_0
\]

and

\[
kA \cos kx_0 - kB \sin kx_0 = kC \cos kx_0 - kD \sin kx_0 + \frac{2m}{\hbar^2} M \Psi(x_0) \hspace{1cm} (A17.2.5)
\]

with \( k = \frac{\sqrt{2mE}}{\hbar} \)

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\(^1\) This boundary condition can easily be generalized for any potential that includes one or more delta functions.
A17.3. Example: tunneling through a delta function

As an example we consider an incoming wave with amplitude 1 incident on the delta function with area, \( M \), and located at \( x = 0 \). The incident, reflected and transmitted waves are then described by:

\[
\Psi_i(x) = \exp ikx = \cos kx + i \sin kx
\]
\[
\Psi_r(x) = r \exp(-ikx) = r \cos kx - ir \sin kx \tag{A17.3.1}
\]
\[
\Psi_t(x) = t \exp(ikx) = t \cos kx + it \sin kx
\]

Where \( r \) is the amplitude of the reflected wave and \( t \) is the amplitude of the transmitted wave. The sum of the incident and reflected wave is the wave function for \( x < 0 \) and the transmitted wave function is the wave function for \( x > 0 \), so that:

\[
A = i(1 - r), B = 1 + r, C = it \text{ and } D = t \tag{A17.3.2}
\]

The boundary conditions at \( x = 0 \) then become:

\[
\Psi(x_0) = 1 + r = t \text{ and } ik(1 - r) =ikt + \frac{2m}{\hbar^2}Mt \tag{A17.3.3}
\]

So that

\[
iki(2 - t) = ikt + \frac{2m}{\hbar^2}Mt \text{ or } t = \frac{ikh^2}{mM + ikh^2} \tag{A17.3.4}
\]

and \( r = t - 1 = \frac{-mM}{mM + ikh^2} \)

Note that both \( r \) and \( t \) are complex numbers, which accounts for a phase shift relative to the incident wave.

The corresponding transmission and reflection are:

\[
T = t t^* = \frac{k^2 \hbar^4}{m^2 M^2 + k^2 \hbar^4} = \frac{2E \hbar^2}{mM^2 + 2E \hbar^2} \tag{A17.3.5}
\]

and \( R = r r^* = \frac{mM^2}{mM^2 + 2E \hbar^2} \)

Confirming that \( T + R = 1 \).