Notes on Analysis of Circuits Containing Mutual Inductance

Equivalent Circuit of Coupled Inductors

When two coils (inductors) are brought into physical proximity with each other, the magnetic field due to current flowing in one of the coils overlaps into the loop formed by the second coil, inducing a voltage in the second coil proportional to the time derivative of the current flowing in the first. We need not concern ourselves with the detailed physics of this process (which is properly the province of a course on quasistatic electromagnetic fields), but will instead present an equivalent circuit model for this coupling which will allow us to analyze circuits in which it occurs.

Two coupled inductors whose inductances are $L_1$ and $L_2$ may possess a significant mutual inductance $M = k \sqrt{L_1 L_2}$, where $k$ is called the coefficient of coupling between the two inductors.\footnote{Capacitors placed in proximity to each other show a similar effect, but it is often of much smaller magnitude and is consequently ignored.} By means of an energy analysis such as that contained in R. E. Thomas, A. D. Rosa and G. J. Toussaint, *The Analysis and Design of Linear Circuits*, 7th ed., sect. 15-3, it can be shown that the values of the coupling coefficient must lie in the range $0 \leq k \leq 1$.

The usual schematic symbol for such coupled inductors is shown in Fig. 1. The dots on one side of each of the inductors serve to indicate the polarity of the coupling—basically which way the field of a positive current in one inductor is oriented relative to a positive current in the other. The meaning of this dot convention is contained in the equivalent circuit shown in Fig. 2. The currents $i_1$ and $i_2$ are taken as the currents whose positive reference directions are into the dotted sides of the inductors $L_1$ and $L_2$ respectively. The dependent voltage sources are connected in series with the inductors as shown, so that their positive reference directions are on the “upper” sides of the circuit, just as the dots are on the “upper” sides of the inductors. All polarities are thus completely and unambiguously determined by this equivalent circuit. For example, applying Kirchhoff’s voltage laws to the left and right sides of the equivalent circuit, we
Figure 2: Equivalent circuit for coupled inductors using dependent sources.

For phasor voltages and currents, we have

\[ v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \]
\[ v_2 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} \]  

(1)

The general rules for analyzing a circuit with mutual inductance are therefore the following:

1) Choose reference directions for the currents \( i_1 \) and \( i_2 \) (or \( I_1 \) and \( I_2 \)) pointing into the dotted side of each inductance.

2) Choose reference polarities for the voltages \( v_1 \) and \( v_2 \) (or \( V_1 \) and \( V_2 \)) following the passive sign convention with respect to the current directions chosen in step 1.

3) Replace the coupled inductors with the dependent-source equivalent circuit of Fig. 2, with the + sides of the dependent sources placed closest to the positive voltage reference polarities of the inductors (i. e., to the dotted sides of the inductors in the original circuit).
Examples

The linear transformer circuit of Fig. 15-13 on page 817 of Thomas, Rosa and Toussaint can be analyzed using our equivalent circuit model, resulting in the circuit of Fig. 3. We will use the node voltage method to find the voltage \( V_D \)

![Circuit Diagram](image)

Figure 3: Equivalent circuit of Fig. 15-13 on page 817 of Thomas, Rosa and Toussaint.

across the load impedance \( Z_L \). Apply KCL to the supernode consisting of nodes B, C and the dependent voltage source \( j\omega M I_2 \):

\[
\frac{V_B - V_A}{Z_S} + \frac{V_C}{j\omega L_1} = 0
\]

while from KCL at the supernode consisting of nodes D, E and the dependent voltage source \( j\omega M I_1 \):

\[
\frac{V_E}{j\omega L_1} + \frac{V_D}{Z_L} = 0
\]

We have the following relations between node voltages:

\[
V_A = V_S; \quad V_B = V_C + j\omega M I_2; \quad V_D = V_E + j\omega M I_1
\]

that allow us to eliminate \( V_A, V_C \) and \( V_E \) from the node-voltage equations

\[
V_B \left( \frac{1}{Z_S} + \frac{1}{j\omega L_1} \right) - \frac{j\omega M I_2}{j\omega L_1} = \frac{V_S}{Z_S}
\]

\[
V_D \left( \frac{1}{Z_L} + \frac{1}{j\omega L_2} \right) - \frac{j\omega M I_1}{j\omega L_2} = 0
\]
Lastly, the currents $I_1$ and $I_2$ can be identified as:

$$I_1 = \frac{V_S - V_B}{Z_S}; \quad I_2 = -\frac{V_D}{Z_L}$$

so we have

$$V_B \left( \frac{1}{Z_S} + \frac{1}{j\omega L_1} \right) + \frac{M}{L_1} \frac{V_D}{Z_L} = \frac{V_S}{Z_S}$$

and

$$V_D \left( \frac{1}{Z_L} + \frac{1}{j\omega L_2} \right) + \frac{M}{L_2} \frac{V_B}{Z_S} = \frac{M}{L_2} \frac{V_S}{Z_S}$$

These equations could now be solved to get $V_B$ and $V_D$.

Suppose now that the position of one of the dots (the one in the secondary) on the transformer was changed, as shown in Fig. 4. Using our equivalent circuit for the coupled inductors, we now have the circuit shown in Fig. 5. Observe that the polarity of the dependent source $j\omega M I_1$ has been reversed, and the reference direction of $I_2$ has been reversed so as to bear the same relationship to the dot as in Fig. 2. When this modified circuit is analyzed by the node voltage method as done above, we get

$$V_B \left( \frac{1}{Z_S} + \frac{1}{j\omega L_1} \right) - \frac{M}{L_1} \frac{V_D}{Z_L} = \frac{V_S}{Z_S}$$

and

$$V_D \left( \frac{1}{Z_L} + \frac{1}{j\omega L_2} \right) - \frac{M}{L_2} \frac{V_B}{Z_S} = -\frac{M}{L_2} \frac{V_S}{Z_S}$$

We see that the effect of moving the dot in this case is to change $M$ into $-M$ in our equations. As long as the polarities of the dependent voltage sources and reference directions for the currents are taken in the same relation to the dots as shown in Fig. 2, this equivalent circuit technique will always give the correct polarities for all quantities involved.
Figure 5: Equivalent circuit of Fig. 4.