\textbf{ILT via Partial Fractions}

\[ X(s) = \frac{1}{(s+1)(s+2)} \]

\[ = \frac{A}{s+1} + \frac{B}{s+2} \]

Multiply through by 
\((s+1)(s+2)\)

\[ 1 = A(s+2) + B(s+1) = (A+B)s + 2A + B \]

Equating coefficients in powers of \(s\):

\[ s: \quad A + B = 0 \quad A = -B \]

\[ 2A + B = 1 \]

\[ A = 1 \quad B = -1 \]

Covers up:

\[ A = (s+1)X(s) \bigg|_{s=-1} = \frac{1}{(s+2)(s+1)} \bigg|_{s=-1} = \frac{1}{-1+2} = 1 = A \]

\[ B = (s+2)X(s) \bigg|_{s=-2} = \frac{1}{s+1} \bigg|_{s=-2} = \frac{1}{-2+1} = -1 = B \]

\[ \text{ILT} \]

\[ x(t) = [1e^{-t} - 1e^{-2t}]u(t) \]

\[ = [e^{-t} - e^{-2t}]u(t) \]
Partial Fractions

\[ F(s) = \sum_{m=0}^{\infty} b_m s^m = \frac{\sum_{n=0}^{N} a_n s^n}{\sum_{n=0}^{\infty} a_n s^n} = \sum_{n=1}^{N} \frac{k_n}{s-p_n} \]

Proper Fraction : \( M < N \)
Improper Fraction : \( M \geq N \)

Must convert improper to proper with long division

\[ F(s) = \frac{2s^3 + 9s^2 + 11s + 2}{s^2 + 4s + 3} \quad M = 3 \quad N = 2 \]

\[ s^2 + 4s + 3 \] \( \underline{\overline{s^2 + 4s + 3}} \) \( 2s^3 + 9s^2 + 11s + 2 \)
\[ 2s^3 + 8s^2 + 6s \]
\[ 2s \]
\[ s^2 + 5s + 2 \]
\[ s^2 + 4s + 3 \]
\[ \text{remainder} \quad s - 1 \]

\[ F(s) = 2s + 1 + \frac{s - 1}{s^2 + 4s + 3} \]

\( M = N \)

\[ F(s) = \sum_{m=0}^{\infty} b_m s^m = \frac{\sum_{n=0}^{N} a_n s^n}{\sum_{n=0}^{\infty} a_n s^n} = \sum_{n=1}^{N} \frac{k_n}{s-p_n} \]

And proceed with cover up as before

Constant term contributes to inverse transform \( \frac{b_n}{a_n} \) \( 8(t) \)
Roots

Quadratic

\[ ax^2 + bx + c = 0 \]
\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Cubic

\[ y^3 + py^2 + qy + r = 0 \]
substitute \( y = x - \frac{p}{3} \)

\[ x^3 + ax + b = 0 \]

\[ a = \frac{1}{3}(3q-p^2) \]
\[ b = \frac{1}{27}(2p^3 - 9pq + 27r) \]

Let

\[ A = \sqrt[3]{-\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \]
\[ B = \sqrt[3]{\frac{-b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \]

Solution of depressed cubic

\[ x_0 = A + B \]
\[ x_1 = -\frac{A+B}{2} + \frac{A-B}{2} \sqrt{-3} \]
\[ x_2 = -\frac{A+B}{2} - \frac{A-B}{2} \sqrt{-3} \]

and solution of cubic are

\[ y_k = x_k - \frac{p}{3} \]

MATLAB

\[ >> \ \text{roots(} [b_m \ b_{m-1} \ldots \ b_1 \ b_0]) \]
Quadratic Factors

Complex conjugate roots will have partial fraction expansion coefficients \( K_1 \) and \( K_2 \) that are complex conjugates, \( K_1 = K_2^* \), giving real combinations as long as coefficients \( a/n \) and \( b/m \) are real. They can thus be combined into a quadratic factor.

\[
F(s) = \frac{4s^2 + 2s + 18}{(s+1)(s^2 + 4s + 13)} = \frac{K_0}{s+1} + \frac{K_1}{s - (-2+3j)} + \frac{K_2}{s - (-2-3j)}
\]

\[
F(s) = \frac{16 - 5^2}{2} = 44 \pm 14^2/4 \cdot 13 = 20 \pm 2 \cdot 3
\]

By method

\[
K_0 = \frac{4s^2 + 2s + 18}{(s+1)(s^2 + 4s + 13)} \bigg|_{s=-1} = \frac{4 \cdot 1 - 2 + 18}{1 - 4 + 13} = \frac{20}{10} = 2
\]

\[
K_1 = \frac{4s^2 + 2s + 18}{(s+1)(s^2 + 4s + 13)} \bigg|_{s=-2+3j} = \frac{4(-5-12j) + (-4+6j) + 18}{(-1+3j)(-2+3j)} = \frac{-5 + 7j(3-j)}{-6j - 18} = \frac{1 + 2j}{3 + j(3-j)} = \frac{1 + 2j}{3^2 + 1^2} = \frac{1 + 2j}{10}
\]

\[
K_2 = K_1^* = 1 - 2j = \sqrt{5} e^{-j63.4}
\]

ILT \( f(t) = \left[ 2e^{-1t} + \sqrt{5} e^{-2t} \left[ e^{j63.4} e^{j3t} + e^{-j63.4} e^{-j3t} \right] \right] u(t) \]

\[
= \left[ 2e^{-t} + 2\sqrt{5} e^{-2t} \cos(3t + 63.4) \right] u(t)
\]

\[
n = 3 = m + 1 \quad K_0 + K_1 + K_2 = 2 + (1+2j) + (1-2j) = 4 = K
\]

\[
m = 2
\]
Alternatively keep quadratic factors

Degrees of freedom in $K_1 = K_2^*$ (real + imaginary part)
can be contained within real $B, C$ in $Bx^2 + C$

$$F(s) = \frac{4s^2 + 2s + 18}{s + 1} \cdot \frac{1}{s^2 + 4s + 13} = A + \frac{Bx + C}{s + 1} + \frac{Dx + E}{s^2 + 4s + 13}$$

clearing factions by multiplying then by denominator

$$4s^2 + 2s + 18 = A(s^2 + 4s + 13) + (Bx + C)(s + 1)$$

$$= (2 + B)x^2 + (8 + B + C)x + (26 + C)$$

Equating powers of $s^2$

$$4 = 2 + B \Rightarrow B = 2$$
$$2 = 8 + B + C \Rightarrow C = 2 - 8 - 2 = -8$$
$$18 = 26 + C \Rightarrow 26 - 8 = 18$$

$$F(s) = \frac{2}{s + 1} + \frac{2s - 8}{(s + 1)^2 + 3^2} = \frac{2}{s + 1} + \frac{2(s + 2)}{(s + 2)^2 + 3^2} = \frac{4}{(s + 2)^2 + 3^2}$$

$$F(t) = \left[2e^{-t} + \frac{2}{\sqrt{10}} \right] u(t)$$

$$r = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5} \quad \theta = \tan^{-1} \frac{1}{2} = 63.4^\circ$$

$$= \left[ e^{-t} + \sqrt{5} e^{-2t} \cos(3t + 63.4^\circ) \right] u(t)$$
Can also solve by putting unknowns on one side

\[
\frac{Bx + C}{s^3 + 4s + 13} = \frac{4s^3 + 2s + 18}{(s+1)(s^2 + 4s + 13)}
\]

\[
\frac{Bx + C}{s+1} = \frac{4s^3 + 2s + 18 - 2(s^2 + 4s + 13)}{(s+1)(s^2 + 4s + 13)} = \frac{2s^3 - 6s - 8}{(s+1)(s^2 + 4s + 13)}
\]

\[
\frac{2s^3 - 6s - 8}{2s^2 + 2s} = \frac{2s - 8}{s^2 + 4s + 8}
\]

Therefore, the constant terms must be equal:

\[
2s - 8 = 0 \Rightarrow s = 4
\]

\[
B = 2 \quad C = -8
\]

or use shortcuts by evaluating both sides at various \(s\) values:

\[
\frac{4s^3 + 2s + 18}{(s+1)(s^2 + 4s + 13)} = \frac{2 + \frac{Bx + C}{s+1}}{s+1} = \frac{2 + \frac{Bx + C}{s^2 + 4s + 13}}{s^2 + 4s + 13}
\]

\[
\text{At } s = 0, \quad \frac{18}{15} = \frac{1}{1} + \frac{C}{15} \Rightarrow C = 18 - 2.13 = -8
\]

\[
\frac{4}{1} = \frac{3}{1} + \frac{B}{1} \Rightarrow \boxed{B = 2}
\]

or can use any other convenient value of \(s\).