ECEN 2260 Circuits as Systems
Spring 2016
Lecture 2

Review Material

Sections 1 through 5 are to be studied thoroughly during the first week of the semester. Section 6 will be discussed during one of the HW review sessions.

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Last revised on 12/26/2015

1. Complex Numbers Review
2. Review of Phasors
3. Dynamical Analogies
4. Additional reviews, examples and tips
5. A RC Time Constant Tutorial
6. P-I Control Tutorial
1. **Complex Numbers Review**

**Fundamentals.**

\[ z = \text{complex number} = x + jy \quad M = \text{Magnitude} = \sqrt{x^2 + y^2} = \|z\| \]

\[ x = \|z\| \cos (\phi) \quad y = \|z\| \sin (\phi) \quad \phi = \tan^{-1}\left(\frac{y}{x}\right) \]

\[ x = \text{Real}(z) \quad y = \text{Imaginary}(z) \quad \phi = \text{Angle}(z) \text{ or Argument}(z) \]

\[ z = x + jy = \text{Rectangular Form} \quad \|z\| (\cos \phi + j \sin \phi) = \text{Trigonometric Form} \]

\[ \|z\| e^{j\phi} = \text{Exponential Form} \quad \|z\| e^{-j\phi} = \text{Polar Form} \]

\[ \bar{z} = z^* = x - jy = \|z\| e^{(-j\phi)} = \|z\| e^{-\phi} = \text{Complex Conjugate} \]

![Diagram of Complex Numbers](image)

**Unit Circle:**

\[ \|e^{j\phi}\| = 1 \quad e^{j\phi} = \phi \]
Addition and Subtraction:

\[ z_1 = x_1 + jy_1 \quad \quad \quad z_2 = x_2 + jy_2 \]

\[ z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \quad \quad \quad z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2) \]

Multiplication and Division:

\[ z_1 z_2 = (x_1 + jy_1)(x_2 + jy_2) = (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1) \]

\[ = \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2} \angle (\phi) \text{, where } \phi = \tan^{-1}\left(\frac{x_1y_2 + x_2y_1}{x_1x_2 - y_1y_2}\right) \]

The above method of multiplication invites algebra errors. In general it is best to convert to polar or exponential form:

\[ z_1 = \|z_1\| \angle (\phi_1) \quad \quad \quad z_2 = \|z_2\| \angle (\phi_2) \]

\[ z_1 z_2 = \|z_1\| \|z_2\| \angle (\phi_1 + \phi_2) = \|z_1\| \|z_2\| e^{j(\phi_1 + \phi_1)} \]

Ditto for division:

\[ \frac{z_1}{z_2} = \frac{\|z_1\|}{\|z_2\|} \angle (\phi_1 - \phi_2) = \frac{\|z_1\|}{\|z_2\|} e^{j(\phi_1 - \phi_1)} \]
Derive trig formulas using Euler’s Expression:

Example:

\[ e^{j\varphi_1} = \cos \varphi_1 + j \sin \varphi_1 \quad e^{j\varphi_2} = \cos \varphi_2 + j \sin \varphi_2 \]

\[ e^{j\varphi_1} e^{j\varphi_2} = e^{j(\varphi_1 + \varphi_2)} = \cos(\varphi_1 + \varphi_2) + j \sin(\varphi_1 + \varphi_2) \quad (1) \]

\[ e^{j\varphi_1} e^{j\varphi_2} = (\cos \varphi_1 + j \sin \varphi_1)(\cos \varphi_2 + j \sin \varphi_2) = \]

\[ e^{j\varphi_1} e^{j\varphi_2} = (\cos \varphi_1 \cos \varphi_2 - \sin \varphi_1 \sin \varphi_2) + j(\sin \varphi_1 \cos \varphi_2 + \cos \varphi_1 \sin \varphi_2) \quad (2) \]

Equate Equations (1) and (2) Real parts:

\[ \cos(\varphi_1 + \varphi_2) = \cos \varphi_1 \cos \varphi_2 - \sin \varphi_1 \sin \varphi_2 \]

Equate Equations (1) and (2) Imaginary parts:

\[ \sin(\varphi_1 + \varphi_2) = \sin \varphi_1 \cos \varphi_2 + \cos \varphi_1 \sin \varphi_2 \]
Roots of Complex Numbers:

Find the roots of \( z = (-1)^{\frac{1}{3}} \) Recall: \( j = e^{i(90\degree + n360\degree)} \)

\[
z = (-1)^{\frac{1}{3}} = (j^2)^{\frac{1}{3}} = j^{\frac{2}{3}} = \left( e^{i(90\degree + n360\degree)} \right)^{\frac{2}{3}}, \quad z = e^{i(60\degree + n240\degree)}, n = 0,1,2,3 \ldots
\]

\[
z_0 = e^{i60\degree} \quad z_1 = e^{i300\degree} = e^{-i60\degree} \quad z_2 = e^{i540\degree} = e^{i180\degree} (= -1)
\]

Application: The three phase voltages and currents in the power distribution grid.
2. **Review of Phasors**

Euler: \( e^{j(\omega t + \varphi)} = \cos(\omega t + \varphi) + j \sin(\omega t + \varphi) \)

The representation of a sinusoid \( v(t) = V_m \cos(\omega t + \varphi) = \text{Re}(V_m e^{j(\omega t + \varphi)}) = \text{Re}(V_m e^{j\varphi} e^{j\omega t}) \) by a complex number \( \vec{V} = V_m e^{j\varphi} \) (or \( \vec{V} \) or \( \vec{V} \) or \( \vec{V} \)) is called a “phasor”.

A phasor transforms back to the time domain by: \( v(t) = \text{Re}(\vec{V} e^{j\omega t}) \)

Phasor analysis is the ideal tool for solving linear circuits in sinusoidal steady state.

Phasor analysis leads to “Impedances”:

\[
\begin{array}{cccc}
R & j\omega L & \frac{1}{j\omega C} \\
\end{array}
\]

**Impedance** \( Z \) is a complex quantity (NOT a phasor) of the form \( Z = R + jX \)
- \( R \) = the real part = Resistance
- \( X \) = the imaginary part = Reactance

\( X \) may be inductive \( (X_L = \omega L) \) or capacitive \( (X_C = -\frac{1}{\omega C}) \) (Note: no \( j \))

All of the Circuits Introduction analysis techniques (series and parallel combinations, voltage and current division, Thevenin and Norton equivalent circuits, superposition, etc.) also apply to phasor analysis.
The voltages generated by our Power Plants are sinusoidal 60 Hz (50 Hz in many other countries) and the impedances of the distribution system can be closely approximated with phasor analysis. (The word “approximated” is used, because our distribution system is not always “steady state” and often includes higher harmonics).

Phasor operation is identical to classical vector operation.

A power distribution grid from the power-plant synchronous generators to the final user (a factory, your home, etc.) consists of impedances of transmission lines, transformers and switching devices. The phasor voltages and currents of this power gridsystem are often represented by a vector diagram that shows the phase differences between the transmission system voltages and currents. (This is a topic covered in ECEN 3170 - Energy Conversion 1).

Review in your text the derivations of:

**Resistor**

\[
\overline{V}_R = \overline{I}_R R
\]

\[
\overline{V}_R = \overline{I}_R R = R
\]

\[
(\mid \overline{I}_R \mid \angle 0^\circ)
\]

**Inductor**

\[
\overline{V}_L = (j \omega L) \overline{I}_L
\]

\[
\overline{V}_L = \omega L \mid \overline{I}_L \mid \angle 90^\circ
\]

\[
\overline{I}_L \text{ leads } \overline{V}_L \text{ by } 90^\circ
\]

**Capacitor**

\[
\overline{V}_C = \frac{1}{j \omega C} \overline{I}_C
\]

\[
\overline{V}_C = \frac{1}{\omega C} \mid \overline{I}_C \mid \angle (-90^\circ)
\]

\[
\overline{I}_C \text{ leads } \overline{V}_C \text{ by } 90^\circ
\]

\[
\overline{V}_R \text{ and } \overline{I}_R \text{ are in-phase}
\]

\[
\overline{I}_L \text{ lags } \overline{V}_L \text{ by } 90^\circ
\]
A non-loop/node equations approach to solving a circuit:

Objective:
1. Look at the circuit by breaking it into functional blocks.
2. Avoid useless manipulations that invite algebra errors.
3. Check units.
4. Substitute numerical values at the very end. You cannot check units after the numbers have been substituted.

Apply these four objectives on the following phasor circuit example:
1. Find the expressions in terms of the circuit components for the input impedance and all steady state voltages and currents (magnitude and phase). Then compute the numerical values.
2. Deduce the value of the input impedance at DC ($\omega \rightarrow 0$) and at very high frequency ($\omega \rightarrow \infty$) by just looking at the circuit; i.e. without calculating any limits.

$v_1(t) = 100 \cos(2000t)$, $L = 250\, \text{mH}$, $C = 0.5\, \mu\text{F}$, $R = 3\, \text{k}\Omega$
Note that $\omega = 2000\, \text{rad/s}$
Solution:

Draw phasor representation of the circuit:

\[ jX_L = j\omega L \]

\[ jX_C = j\frac{-1}{\omega C} \]

Note:
\( X_L = \text{Inductive Reactance} = \omega L \text{ (no } j\text{)} \)
\( X_C = \text{Capacitive Reactance} = -\frac{1}{\omega C} \text{ (no } j\text{)} \)
1. Functional “blocks”:

a) Input Impedance:
   - Impedance of L in series with impedances of C parallel with R.

b) Voltage divider: Impedance of C parallel with R over the input impedance.

c) Current divider between C and R.

\[ Z_{\text{in}} = jX_L + R || jX_C \]
\[ I_L = \frac{V_1}{Z_{\text{in}}} = \frac{1}{jX_L + R || jX_C} \]

b) Voltage divider:
\[ V_2 = \frac{V_1 R || jX_C}{jX_L + R || jX_C} \]

c) Current dividers:
\[ I_C = \frac{I_L R}{R + jX_C}, \quad I_R = \frac{jX_C}{R + jX_C} \quad (= I_L - I_C) \]

2. Avoid useless manipulations:
   The above results look understandable, tidy and uncluttered; so no further “simplification” manipulations are necessary.

3. Check Units:
   Verify left side vs. right side units. Convince yourself.

4. Substitution of numerical values.
   We now need to do some numerical calculations. Even though there are many computer based computation tools available to us, we still need to be able to make computations (estimations) by hand.

   Why?

   We must be able to verify (estimate) the computer output results. So if your computer program computes an output current of 10 amps at an output voltage of 100 V for an LF356 op-amp, then you should recognize that this is not possible. Maybe it should be 10 mA at 10 V. Just recall that “garbage-in = garbage-out”.
Compute the reactance values:

\[ R = 3000 \, \Omega \]
\[ X_L = \omega L = (2000)(250)10^{-3} = 500 \, \Omega \]
\[ X_C = -\frac{1}{\omega C} = -\frac{1}{(2000)(0.5)10^{-6}} = -1000 \, \Omega \]

Compute the different impedance values:

\[ \frac{R||jX_C}{R + jX_C} = \frac{jRX_C}{3000 - j1000} = \frac{3000(-j1000)}{3000 - j1000} = \frac{-300j}{3 - j} \quad \frac{3 + j}{3 + j} = \frac{-9000j + 3000}{10} \]

\[ = 300 - 900j \, \Omega \]

Input Impedance: \[ Z_{in} = jX_L + R||jX_C = j(500 + 300 - 900j) = 300 - 400j \, \Omega \]

\[ \frac{R||jX_C}{jX_L + R||jX_C} = \frac{300 - 900j}{300 - 400j} = \frac{3 - 9j}{3 - 4j} \quad \frac{3 + 4j}{3 + 4j} = \frac{9 + 12j - 27j + 36}{9 + 16} \]

\[ = \frac{45 - 15j}{25} = \frac{9 - 3j}{5} \, \Omega \]

\[ \frac{jX_C}{R + jX_C} = \frac{(-j1000)}{3000 - j1000} = \frac{-j}{3 - j} \quad \frac{3 + j}{3 + j} = \frac{-3j + 1}{10} = 0.1 - 0.3j \, \Omega \]
Compute the current and voltage phasors:

\[
\tilde{I}_L = \frac{\tilde{V}_1}{Z_{in}} = \frac{1}{jX_L + R||jX_C} = \frac{100\angle0^\circ}{300 - 400j} = \left(\frac{100\angle0^\circ}{500\angle-53.1^\circ}\right) = 0.2\angle53.1^\circ \text{ A}
\]

\[
\tilde{I}_C = \tilde{I}_L \frac{R}{R+jX_C} = \left(0.2\angle53.1^\circ\right)\frac{3000}{3000-j1000} = \left(0.2\angle53.1^\circ\right)\frac{3}{3-j3+J} = \left(0.6\angle53.1^\circ\right)^{3+J} = \left(0.06\angle53.1^\circ\right)(3+j) = \left(0.06\angle53.1^\circ\right)(\sqrt{10}\angle18.4^\circ) = 0.19\angle71.5^\circ \text{ A}
\]

\[
\tilde{I}_R = \tilde{I}_L - \tilde{I}_C = 0.2\angle53.1^\circ - 0.19\angle71.5^\circ \text{ This method of computing } \tilde{I}_R \text{ looks cumbersome as we will need to convert from polar to rectangular and back to polar forms. So let us use the result of the current division:}
\]

\[
\tilde{I}_R = \tilde{I}_L \frac{jX_C}{R + jX_C} = \left(0.2\angle53.1^\circ\right)(0.1 - 0.3j) = \left(0.2\angle53.1^\circ\right)(0.316\angle-71.5^\circ) = 0.063\angle-18.4^\circ \text{ A}
\]

\[
\tilde{V}_2 = \tilde{V}_1 \frac{R||jX_C}{jX_L + R||jX_C} = \frac{9 - 3j}{5} \left(100\angle0^\circ\right) = 60(3-j) = (60)(\sqrt{10}\angle-18.4^\circ) = 189\angle-18.4^\circ \text{ V}
\]

1. Check for yourself the results by computing the output voltage with \( \tilde{V}_2 = \tilde{I}_R R \) and \( \tilde{V}_2 = \tilde{I}_C (jX_C) \)
2. Draw a phasor diagram showing all phasor currents and voltages. Such a diagram may be on a test/quiz.
3. The above showed a computational methodology. So numbers to three decimals were used. Often this accuracy is not needed. What will be needed is for you to be able to make an approximation of a RELATIVE value of a variable/component with respect to other variables/components. So go back to the above circuit and see how close you can estimate the final
results. Sketching a phasor-diagram (vector diagram) is a very useful tool for this.

Behavior of circuit near DC:

\[ jX_L = j\omega L \]

\[ jX_C = j\left(-\frac{1}{\omega C}\right) \]

Note:
\( X_L = \text{Inductive Reactance} = \omega L \) (no j)
\( X_C = \text{Capacitive Reactance} = -\frac{1}{\omega C} \) (no j)

\[ \omega \to 0: \]
\[ j\omega L \to \text{“short circuit”} \]
\[ \frac{1}{j\omega C} \to \text{“open circuit”} \]
Thus:
\[ ||Z|| \to R \text{ and } \angle Z \to 0^\circ \]

Behavior of circuit at high frequency:

\[ jX_L = j\omega L \]
\[ jX_C = j\left(-\frac{1}{\omega C}\right) \]

Note:
\( X_L = \text{Inductive Reactance} = \omega L \) (no j)
\( X_C = \text{Capacitive Reactance} = -\frac{1}{\omega C} \) (no j)

As \( \omega \to \infty \): \( j\omega L \to \infty \)
\[ \frac{1}{j\omega C} \to \text{“short circuit”} \]
\[ Z_{\text{in}} = j\omega L + R\left|\frac{1}{j\omega C}\right| \]

Since \( j\omega L \to \infty \), the inductor dominates. Thus:
\[ ||Z|| \to \omega L \to \infty \text{ and } \angle Z \to 90^\circ \text{ since } \angle \frac{j\omega L}{90^\circ} \]
For the above circuit problem we were asked to compute all phasor voltages and currents. We obtained the answers with a minimum of “clutter” by using the inherent functionality of the circuit.

Suppose we were asked to show how the magnitude and phase of the input impedance depends on $\omega$. We would then have no choice but doing some “ugly” complex algebra manipulations. However note how we can minimize the “ugliness” by using polar multiplication/division as indicated earlier.

$$Z_{\text{in}}(j\omega) = j\omega L + \frac{R}{j\omega C} = j\omega L + \frac{R}{1 + j\omega RC}$$

$$= \frac{j\omega L + (j\omega L)(j\omega RC) + R}{1 + j\omega RC} = \frac{(R - \omega^2 RLC) + j(\omega L)}{1 + j(\omega RC)} = \frac{\text{num}}{\text{denom}}$$

Magnitude:

$$||Z_{\text{in}}(j\omega)|| = \frac{||\text{num}||}{||\text{denom}||} = \frac{\sqrt{(R - \omega^2 RLC)^2 + (\omega L)^2}}{\sqrt{1 + (\omega RC)^2}}$$

Phase:

$$\angle Z_{\text{in}}(j\omega) = \angle \frac{\text{num}}{\text{denom}} = \angle \text{num} - \angle \text{denom} \implies$$

$$\angle Z_{\text{in}}(j\omega) = \tan^{-1}\left(\frac{\omega L}{(R - \omega^2 RLC)}\right) - \tan^{-1}\left(\frac{\omega RC}{1}\right)$$

These results show how the impedance magnitude and phase are somewhat complicated functions of $\omega$. We will be addressing this topic very extensively during this semester.
3. **Dynamical Analogies**

Ref.: Dynamical Analogies by Harry F. Olson, 2\(^{nd}\) edition, 1943/1958, free down loadable. Highly Recommended.

A **mass** \( M \) moving at **velocity** \( u \) due to **force** \( F \)

\[
\begin{array}{c}
\text{M} \\
\text{F} \\
u \text{ (velocity)}
\end{array}
\]

Newton’s Law

\[
F = M \frac{du}{dt}
\]

A **compliance** \( C \) (= spring \( K_s \)) compressed by **distance** \( x \) due to **force** \( F \)

\[
\begin{array}{c}
\text{F} \\
x \\
K_s = \frac{1}{C_{\text{compliance}}}
\end{array}
\]

\[
dF = K_s \, dx = \frac{1}{C} \, dx \Rightarrow dx = C \, dF \Rightarrow dx = \frac{dF}{C} \Rightarrow \\
\frac{dx}{dt} = \frac{dF}{Cdt} \Rightarrow \\
u = \frac{dF}{Cdt}
\]

A **force** \( F \) due to a **friction** \( B \) as a result of an object moving at **velocity** \( u \)

\[
\begin{array}{c}
\text{F} \\
\text{B}
\end{array}
\]

\[
F = Bu
\]

An **inductor** \( L \) with a **current** \( i \) due to **voltage** \( v \)

\[
\begin{array}{c}
\text{v} \\
i \\
L
\end{array}
\]

\[
v = L \frac{di}{dt}
\]

A **capacitor** \( C \) charged by **charge** \( q \) due to **voltage** \( v \)

\[
\begin{array}{c}
\text{v} \\
i \\
C
\end{array}
\]

\[
i = C \frac{dv}{dt}
\]

A **voltage** \( v \) across a **resistance** \( R \) due to a **current** \( i \)

\[
\begin{array}{c}
\text{v} \\
i \\
R
\end{array}
\]

\[
v = Ri
\]
4. **Additional reviews, examples and tips**

**A. Approximations**

Often a circuit can be simplified by looking at approximations. For example: A 10 ohm resistor in parallel with a 1000 ohm resistor still “equals” 10 ohms; especially when the resistors have 10% tolerances.

Initial approximations may reduce a complicated looking circuit into some very elementary functional blocks. This in turn may turn a “spaghetti” looking “mess” into a very logical and readable diagram.

**B. Units**

What is wrong with the following equation?

\[
I_2 = \frac{j\omega I_1 - V_1}{V_1 + R + j\omega L + \frac{1}{j\omega C}} \text{ A}
\]

This equation is complete nonsense. Volts are added to ohms!!!!!!!!!!!!

You can catch a majority of the errors with units-expressions. At one time in the past we (the USA tax payers) lost a space craft because of an error in units.

Note: Suppose during an exam you discover that the units don’t mach and no time to fix it. If you simply note that you are aware of this error, then I will be much more lenient with the grading.
C. Circuit Manipulation

A resistance $R_1$ in series with a parallel combination of $R_2$ and $R_3$:

\[
R_1 \quad R_2 \quad || \quad R_3
\]

The equivalent resistance is $R_{eq} = R_1 + R_2 || R_3$

$R_2 || R_3 = \text{parallel combination of } R_2 \text{ and } R_3 = \frac{1}{\frac{1}{R_2} + \frac{1}{R_3}} = \text{inverse addition}$

We could “simplify” as follows:

\[
R_{eq} = R_1 + R_2 || R_3 = R_1 + \frac{R_2 R_3}{R_2 + R_3} = \frac{R_1 R_2 + R_1 R_3 + R_2 R_3}{R_2 + R_3}
\]

However, even though the manipulation is correct, it is useless and is an opportunity to make algebra errors.
The original form $R_{eq} = R_1 + R_2 || R_3$ is much more understandable.
The $||$ operator can be viewed as inverse addition and is an operator similar to “+”, “-“, etc.

Computation of $R_{eq}$ is actually easier using the original expression:

$R_{eq} = R_1 + R_2 || R_3 = 470\,\Omega + 100\,\Omega || 1\,k\Omega = 470\,\Omega + 91\,\Omega = 561\,\Omega$
D. Some more phasor circuits using functionality to solve them

For each of the following circuits, write an expression for the output voltage phasor $\vec{V}_{\text{out}}$ in terms of the input voltage phasor $\vec{V}_{\text{in}}$ and component values. Do this by inspection without resorting to loop/node equations.

1. An Op-Amp amplifier

![Inverting Amplifier Diagram]

Inverting Amplifier Expression:

$$\vec{V}_{\text{out}} = \vec{V}_{\text{in}} \left(-\frac{R_2}{R_1}\right)$$

2. Another Op-Amp amplifier

![Inverting Amplifier with Series and Parallel Impedances Diagram]

Inverting Amplifier Expression with series and parallel impedances:

$$\vec{V}_{\text{out}} = \vec{V}_{\text{in}} \left(-\frac{R_2}{R_1} \left(\frac{1}{j\omega C_2}\right)\right)$$

$$\frac{1}{R_1 + \left(\frac{1}{j\omega C_1}\right)}$$

Can multiply out into polar numerator and denominator forms as before to find amplifier magnitude and phase as function of $\omega$. 
3. A voltage divider

Use voltage divider expression:

\[
\vec{V}_{out} = \vec{V}_{in} \frac{R_2 \| \frac{1}{j\omega C}}{R_1 + R_2 \| \frac{1}{j\omega C}}
\]
4. A more complicated voltage divider

Use voltage divider expression twice:

\[ V_{\text{out}} = V_{\text{in}} \left( \frac{1}{j\omega C_1} \parallel \left( \frac{R_2 + 1}{j\omega C_2} \right) \right) \left( \frac{1}{R_2 + \frac{1}{j\omega C_2}} \right) \]

The first bracketed term divides the input voltage into the voltage across \( C_1 \) including the loading of \( R_2 \) and \( C_2 \). The second bracketed term divides the voltage across \( C_1 \) into the output voltage.

Another approach to solving this circuit would be using the Thevenin equivalent for the input circuit \( R_1-C_1-V_{\text{in}} \). Then combine the Thevenin equivalent circuit with \( R_2-C_2 \), followed by voltage division to find \( V_{\text{out}} \). You should do this for yourself; this could be on a quiz.
Suppose now that the magnitude of the $R_2$-$C_2$ impedance is much greater than the magnitude of the $C_1$ impedance; i.e. a “large” impedance in parallel with a “small” impedance.

Recall the earlier statement:
A 10 ohm resistor in parallel with a 1000 ohm resistor “is” still 10 ohms; especially if the resistors have 10% tolerances.

So in our case here we may be able to approximate the final result by ignoring the loading impedance of $R_2$-$C_2$.

The use of the parallel symbols $\parallel$ in the first bracketed term makes it pretty clear that we can simply omit the second term $(R_2 + \frac{1}{j\omega C_2})$ in both the numerator and denominator. So in that case

\[
\vec{V}_{\text{out}} \approx \vec{V}_{\text{in}} \left[ \frac{1}{j\omega C_1} \right] \cdot \left[ \frac{1}{R_1 + \frac{1}{j\omega C_1}} \right] \cdot \left[ \frac{1}{R_2 + \frac{1}{j\omega C_2}} \right]
\]

This method of approximation would have been practically impossible if we would have “simplified” the expressions by multiplying out the parallel terms. Again, the moral should be clear: Show functional blocks inside your equations whenever and wherever possible.
Let us now suppose that the above circuit is a simple low voltage circuit on your lab bread-board and let us further suppose that now the magnitude of the $R_2-C_2$ impedance is at the same order of magnitude as the $C_1$ impedance. So we no longer can ignore the loading by $R_2-C_2$. How could we change the circuit so that we could still ignore the loading?

Recall the voltage-follower from Circuits 1:

$$\begin{align*}
V_{out} & \approx V_{in} \left[ \frac{1}{\frac{j\omega C_1}{R_1} + 1} \right] \cdot \left[ \frac{1}{\frac{j\omega C_2}{R_2} + 1} \right]
\end{align*}$$

Before you started Circuits 1, this schematic might have looked like a bowl of spaghetti. Now you should be able to see the three functional blocks: The $R_1-C_1$ impedance, the voltage follower and the $R_2-C_2$ impedance.

You should write out for yourself the equations that lead to the final input to output expression:
E. A first order transient problem

\begin{center}
\begin{circuitikz}

\draw (0,0) node [ground] (g) {} -- (0,-0.5) node [ground] (n) {} -- (0,-1) node [voltage source, right] (v) {\( v_{\text{in}}(t) \)} -- (0,-1.5) node [capacitor, right] (c) {\( C \)} -- (1,-2) node [voltage sink, right] (v2) {\( v_{\text{out}}(t) \)} -- (1,-2.5) node [ground] (n2) {} -- (1,-3) node [ground] (g2) {} -- (0,-3) node [ground] (g3) {} -- (0,-3.5) node [ground] (n3) {} -- (0,-4) node [ground] (n4) {} -- cycle;
\draw (0,-2) node [resistor, right] (r) {\( R_1 \)} -- (0,-2.5) node [ground] (g1) {} -- (0,-3) node [ground] (g2) {} -- cycle;
\end{circuitikz}
\end{center}

Objective:
For \( v_{\text{in}}(t) \) equals a step at \( t=0 \) of magnitude \( V_P \) volts, sketch \( v_{\text{out}}(t) \) and \( i_c(t) \) and explain physically what happens, without formally solving the differential equation.

Solution:
In equilibrium the cap behaves as an open circuit. Hence:

For \( t < 0 \), \( v_{\text{out}}(t) = v_{\text{in}}(t) = 0 \) and \( i_c(t) = 0 \)

For \( t \to \infty \), \( v_{\text{out}}(t) \to v_{\text{in}}(t) = V_P \) and \( i_c(t) = 0 \)

Recall from Circuits 1 that the variables in these types of circuits progress as exponentials with a certain time-constant.
So in this circuit it stands to reason that the output voltage increases with an exponential from 0 volt at \( t=0 \) to \( V_P \) volts at \( t \to \infty \). The only time constant possible is \( \tau = RC \) sec.
Now let us look at the current. Recall that the charge \( q \) on the cap plates is directly proportional to the voltage across the cap, where \( C \) is the proportionality constant: \( q_C(t) = Cv_{\text{out}}(t) \). To change the cap’s charge, a current must flow through the cap \( i_C(t) = \frac{dq}{dt} \).

But in this circuit \( i_C(t) \) is given by \( i_C(t) = \frac{v_{\text{in}}(t) - v_{\text{out}}(t)}{R} \). In words: the voltage across the resistor determines the capacitor current.

For \( t < 0 \), \( v_{\text{out}}(t) = v_{\text{in}}(t) = 0 \), \( i_C(t) = 0 \) → The cap’s voltage does not change, because the current is zero; i.e. \( v_{\text{out}}(t) = 0 \).

For \( t = 0^+ \), \( v_{\text{in}}(0^+) = V_P, v_{\text{out}}(0^+) = 0 \). → \( i_C(0^+) = \frac{V_P}{R} \) → The capacitor charge \( q_C(t) \) increases → \( v_{\text{out}}(t) \) increases.

As \( v_{\text{out}}(t) \) approaches \( v_{\text{in}}(t) = V_P \), the voltage across R decreases → \( i_C(t) \) decreases → the capacitor voltage \( v_{\text{out}}(t) \) increases more slowly.

For yourself you should draw a sequence of plots for each of the above steps. The end result will look like:

The actual equations are:

\[
v_{\text{out}}(t) = 0 \text{, for } t \leq 0, \quad v_{\text{out}}(t) = V_P \left( 1 - e^{-\frac{t}{\tau}} \right) \text{, for } t \geq 0
\]

\[
i_C(t) = 0 \text{, for } t \leq 0, \quad i_C(t) = \left( \frac{V_P}{R} \right) e^{-\frac{t}{\tau}} \text{, for } t \geq 0 \text{, with } \tau = RC.
\]
We obtained the previous results by simple logical reasoning using fundamental electrical properties, without ever writing any node/loop equations. You should strive to do this for each problem you are trying to solve. In other words: look at the problem for a while and “absorb” its structure into your mind. Then, when you do write equations, you have a better chance to see if the results make physical sense. Last but not least: Check your units.

F. Another first order transient circuit problem

![Circuit Diagram]

For this circuit “F”, do for yourself the reasoning of circuit “E”. I could put this exact circuit “F” on a quiz. So work together with a few classmates and really get to know the above described scenario of functional reasoning.

5. A RC Time Constant Tutorial

\[ R \frac{di(t)}{dt} + N_0 i(t) = N_0 (t) \]
\[ i_i(t) = C \frac{dv(t)}{dt} \]
\[ N_{0e} = N_{0n} (t) + N_{0f} (t) \]
\[ = \text{natural response} + \text{forced response} \]

**Solution:**
\[ N_{0e} = N_{0n} (t) + N_{0f} (t) \]
\[ = \text{natural response} + \text{forced response} \]

**Due to Circuit:**
**Due to Input**

First solve homogeneous equations:
\[ RC \frac{du(t)}{dt} + u(t) = 0 \]
Assume \[ u_n = k_1 e^{st} \] \[ \frac{du_n}{dt} = k_2 e^{st} \]
\[ \Rightarrow RC k_2 e^{st} + k_2 e^{st} = 0 \]
\[ s = -\frac{1}{RC} \]
\[ \Rightarrow u_n = k_1 e^{-\frac{t}{RC}} = \text{natural response} \]
FORCED RESPONSE:

Assume \( N_v(t) = k_2 \Rightarrow \frac{dN_v(t)}{dt} = 0 \)

Substitute into

\[
RC \frac{dN_v(t)}{dt} + N_v(t) = V_A
\]

\( 0 + k_2 = V_A \Rightarrow V_A = k_2 \)

TOTAL RESPONSE:

\[
N_v(t) = N_v(0) + N_v(0)e^{-\frac{t}{RC}} = V_A \left(1 - e^{-\frac{t}{RC}}\right)
\]

Apply initial condition: \( N_v(0) = 0 \)

\[ N_v(0) = k_2 + V_A = 0 \Rightarrow k_2 = -V_A \]

\[ N_v(t) = V_A - V_A e^{-\frac{t}{RC}} \Rightarrow N_v(t) = V_A \left(1 - e^{-\frac{t}{RC}}\right) \]

\[ N_v(t) = V_A \left(1 - e^{-\frac{t}{RC}}\right), \quad T_c = RC = \text{time constant} \]

\[ 0.37 \approx \frac{1}{3}, \quad 0.63 \approx \frac{2}{3} \]

Note that \( 0.37 \approx \frac{1}{3}, \quad 0.63 \approx \frac{2}{3} \)
Look at $T_c = RC$ for varying $R \neq C$:

If $R$ and/or $C$ increase $\Rightarrow T_c$ increases

$V_a$ $0.63$

$\text{Increasing time constant}$

$T_c_1 < T_c_2 < T_c_3 < T_c_4 < T_c_5 < T_c_6$

"Longer" time constant

Means "longer" time to reach steady state:

$N_{cp}(t) = N_{cs}(t) = V_a$

The above discussed the time constant for an "RC" circuit.

The same analysis hold for say an "BJT" circuit; a rotational dynamics "circuit". (See next page)
**BUT** \( \text{TIME CONSTANT} \)

APPLIED TORQUE TO A ROTATING CO-ORD:

\[
T(t) = J \dot{\omega}(t) + B \omega(t) \quad (\text{NEWTON'S LAW})
\]
\[\omega(0) = 0\]

\[
T(t) = T_a \cdot \text{UNIT STEP}
\]

\[
\Rightarrow 0
\]

\[
J \frac{d\omega}{dt} + B \omega = T_a, \quad t > 0
\]

**SIMILAR SOLUTION AS FOR LC CIRCUIT**:

\[
\omega(t) = \frac{T_a}{B} \left[ 1 - e^{-t/\tau} \right] \quad \text{RAD/SEC}
\]

OR \[
\omega(t) = \frac{T_a}{B} \left[ 1 - e^{-t/\tau_c} \right], \quad \tau_c = \frac{J}{B} \quad \text{TIME CONSTANT}
\]

\[
\tau_c = \frac{J}{B} \Rightarrow \text{INCREASE J} \Rightarrow \text{LARGER TIME CONSTANT}
\]
\[
\tau_c \Rightarrow \text{DECREASE B} \Rightarrow \text{"1"}
\]

**THIS RESULT SHOULD AGREE WITH YOUR INTUITION.**

**SIMILAR TIME CONSTANTS APPEAR FOR MECHANICAL TRANSLATION, THERMAL SYSTEMS AND OTHERS**
6. P-I Control Tutorial

**P-I Control**

\[ G_c(s) \]

\[ R(s) \]

\[ E(s) \]

\[ K_1 \]

\[ X(s) \]

\[ \frac{R}{s+R} \]

\[ Y(s) \]

\[ K_1, K_2, R \text{ are positive values} \]

\[ R(s), E(s), X(s), Y(s) = \mathcal{L}\text{ of } R(t), E(t), X(t), Y(t) \]

\[ G_A(s) = \text{"Plant" = System to be controlled} \]

\[ G_c(s) = \text{Controller} = K_1 + \frac{K_2}{s} = P-I \text{ controller} \]

\[ G_{cl} = \text{Open Loop Transfer Function} = G_c(s) \cdot G_A(s) \]

\[ R(s) = \text{Desired Setpoint (Command)}, \text{ we will use } A(t) = A(t) = \text{step} = R(s) = \frac{1}{s} \]

\[ Y(s) = \text{System output that needs to follow } R(s) \text{ as close as possible, with } y(\infty) = 1 \]

\[ \Rightarrow e(\infty) = 0 \]

\[ \text{I of IR} \]
FROM BLOCK DIAGRAM:
\[ Y = G_{oc} \quad E = G_{oc} (R-Y) = G_{oc} R - G_{oc} Y \Rightarrow \]
\[ \Rightarrow Y(1+G_{oc}) = G_{oc} R \Rightarrow Y = \frac{G_{oc}}{1+G_{oc}} R \]

OR \[ Y = G_{ac} R \], WHERE

\[ G_{ac} = \text{CLOSED LOOP TRANSFER FUNCTION} = \frac{G_{oc}}{1+G_{oc}} \]

\[ G_{oc} = \frac{G_{c}(s) \cdot G_{o}(s)}{G_{o}(s)} = \left( k + \frac{k_2}{s} \right) \frac{1}{s+\alpha} = \]

\[ G_{ac} = \frac{k_1 R}{s+\alpha} + \frac{k_2 R}{s(s+\alpha)} = \frac{k_1 R(s+k_2/s)}{s(s+\alpha)} \]

\[ \text{S-PLANE} \]

\[ 20F \]
Find closed loop transfer function:

\[
G_{cl} = \frac{G_{oc}}{1 + G_{oc}} = \frac{\frac{KR}{S(S + 1)}}{1 + \frac{KR}{S(S + 1)}} = \frac{KR(3 + k_2/k_1)}{S(S + 1) + KR(3 + k_2/k_1)}
\]

\[
G_{cl} = \frac{KR}{S^2 + S(1 + k_2/k_1) + k_2k_1} = \frac{KR(3 + k_2/k_1)}{S^2 + 2y \omega_b S + \omega_b^2}
\]

Note the 2nd order denominator.

Look at system output \( y(s) \):

\[
y(s) = G_{cl}(s) \cdot r(s) \quad \int \Rightarrow y(s) = \frac{G_{oc}(s)}{s}
\]

We want to know the final error between the input \( r(t) \) and output \( y(t) \)

\[
e(t) = r(t) - y(t)
\]

We like this to equal zero for \( t \to \infty \). This means that \( y(t) \) for \( t \to \infty \) needs to equal \( r(t) = 1 \)

Find \( y(t) \) as \( t \to \infty = y(\infty) \) by using

the final value theorem

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\[ y(t) \big|_{t \to \infty} = y(0) = \lim_{s \to 0} sY(s) = \]
\[ = \lim_{s \to 0} s \frac{G_0(s)}{s} = \lim_{s \to 0} G_0(s) = \]
\[ = \lim_{s \to 0} \left[ k_2 \frac{5s + k_2/k_1}{5^2 + 5(1+k_1)s + k_2} \right] = \]
\[ = \frac{0 + k_2 \left( \frac{k_2}{k_1} \right)}{0 + 0 + k_2} = \frac{k_2 k_2}{k_1 k_1} = 1 \Rightarrow \]
\[ \Rightarrow y(\infty) = 1 \]

\[ e(\infty) = r(\infty) - y(\infty) = 1 - 1 = \text{ZERO} \]

This zero error is a result of the integral control \( \frac{1}{s} \).

Now convince yourself by looking at the original block diagram.

Suppose \( y(t) < \text{set} \) at some time \( t \).

Then the block \( \frac{y_0}{s} \) will continue.
TO INTEGRATE \( y(t) \) WILL BE INCREASING
\( \Rightarrow \) \( x(t) = 1/2(x(t) - y(t)) \) WILL CONTINUE TO
DECREASE, ETC.

THE DYNAMICS BETWEEN \( t = 0 \) AND
\( t \to \infty \) IS DETERMINED BY THE
THE CLOSED LOOP TRANSFER FUNCTION
POLES AND ZEROS

\[ G_{cl} = \frac{s + k_1/k_2}{s^2 + s(1 + k_2) + k_2} = \frac{s + k_1/k_1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]

A COMPLEX PAIR HAS BEEN SHOWN HERE.
THEY COULD OF COURSE BE SIMPLE REAL
POLES AS WELL... SO HOW DO WE
DETERMINE \( k_1, k_2 \) THAT WILL GIVE
(CIS THE DESIRED CLOSED LOOP POLES?)

SOFIR
IN FACT: WHAT DO WE MEAN BY "DESIRED" CLOSED LOOP ROCES?

THE ANSWER TO THESE QUESTIONS ARE FOR FUTURE COURSES SUCH AS ECEN 4138 (CONTROL SYSTEMS). FOR HERE I WILL ONLY SUGGEST TO USE \( y = 0.707 \) AS A STARTING POINT (TO BE TALKED ABOUT DURING LECTURE).

IN FUTURE COURSES YOU WILL ALSO LEARN TO FIND THE CLOSED LOOP ROCES FROM THE ROOT LOCUS OF THE OPEN LOOP TRANSFER FUNCTION \( G(s) \).

THIS "LOCUS" SHOWS HOW THE CLOSED LOOP ROCES WILL VARY AS FUNCTION OF A VARYING PARAMETERS SUCH AS \( K_1 \) OR \( K_2 \).

MATLAB HAS A BUILD-IN "LOCUS" SCRIPT.
WHY NOT JUST KEEP THINGS SIMPLE WITH PROPORTIONAL CONTROL ONLY AND NO INTEGRAL ACTION (⇒ K₂ = 0)?

Then \( G_c = \frac{k_1 R}{s + a} \)

\( G_{cl} = \frac{G_c}{1 + G_c} = \frac{k_1 R}{s + k_1 R} = \frac{k_1 R}{s + a(1 + k_1)} \)

\( Y(s) = G_{cl} (s) R(s) = \frac{k_1 R}{s + a(1 + k_1)} \)

\( Y(s) = \frac{k_1 R}{s + a(1 + k_1)} \) \( \frac{1}{s} \) \( ⇒ \)

\( Y(s) = \lim_{s \to 0} \frac{k_1 R}{s + a(1 + k_1)} \) \( \frac{1}{s} \) \( = \)

\( = \lim_{s \to 0} \frac{k_1 R}{s + a(1 + k_1)} = \frac{k_1 R}{a(1 + k_1)} = \frac{k_1}{1 + k_1} \)

OR \( Y(s) = \frac{1}{1 + \frac{1}{k_1}} \). WE NEED \( Y(s) \) TO FOLLOW \( R(s) = M \) \( ⇒ sR_p = 1 \) \( (s \to 0) \)

Thus, we want \( Y(0) = 1 \). This can only happen if \( K = 0 \). This of course——
is not possible.

So: \( y(\infty) = \frac{1}{1 + \frac{1}{k_1}} \)

The error @ \( t \to \infty \) is

\[
e(\infty) = \eta(\infty) - y(\infty) = 1 - \frac{1}{1 + \frac{1}{k_1}}
\]

\[
= \frac{1 + \frac{1}{k_1} - 1}{1 + \frac{1}{k_1}} = \frac{\frac{1}{k_1}}{1 + \frac{1}{k_1}}
\]

\[e(\infty) = \frac{1}{1 + k_1}\]

Example: Suppose the max value of \( k_1 = 9 \) (could be the gain of an op-amp). Then

\[
e(\infty) \bigg|_{k_1=9} = \frac{1}{1+9} = 0.1 = 10\%
\]

An integral action (\( k_2 \neq 0 \)) can be used to get \( e(\infty) = 0 \)

\[8.0 \equiv 12\]
**EXAMPLE: DC MOTOR SPEED CONTROL**

**MOTOR DYNAMICS**

1. \[ T_e = T_m + J \omega_m + B \omega_m \] (Newton's law)

2. \[ T_e = K_m I_e \] = Applied torque by motor armature
   
   \[ K_m = \text{Motor torque constant (also back EMF constant)} \]
   
   \[ I_e = \text{Armature current} \]

   \[ T_m = \text{Torque that does useful mechanical work} \]

   \[ J = \text{Polar moment of inertia (motor and load)} \]

   \[ \omega_m = \text{Motor rotational speed} \]

   \[ B = \text{Viscous friction constant} \]

   **The voltage equation of the motor armature is**

3. \[ V = R_a I_e + L_a \frac{dI_e}{dt} + K_m \omega_m \] (see circuit below)

   \[ R_a = \text{Armature resistance} \]

   \[ L_a = \text{Inductance} \]

   \[ V = \text{Applied motor voltage} \]

---

**Diagram:**

- A circuit diagram showing the connections and variables involved in the motor speed control system.

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TAKE LA Place TRANSFORM OF

(1), (2), (3):

(1) \[ T_a = T_m + JS \omega_m + B \omega_m \]

(2) \[ T_a = k_m I_a \]

(3) \[ V = R_e I_a + 5a I_a + k_m \omega_m \]

LET US PUt THESE IN A BLOCK DiAGRAM:

(1) \[ JS \omega_m = T_a - T_m - B \omega_m \]

(2) \[ I_a \rightarrow \frac{k_m}{T_a} \]

(3) \[ 5a I_a = V - R_e I_a - k_m \omega_m \]
Put all three together:

Very often for initial analysis we may let $L \to 0$ and ignore $k_m w_m$ then equation (3) reduced to:

(3) $\Rightarrow V = R_e I_a$

Assume for now also that $T_m < \frac{J}{2}$.

Now our simplified block diagram is:

\[ \frac{W_m}{T_a} = \frac{G_m}{1 + B G_m} = \frac{1/55}{1 + 8/55} = \frac{1}{J5 + B} = \frac{1/5}{5 + 8/5} \]

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This reduces the block diagram to

\[ V \xrightarrow{K_M/2a} \frac{U/s}{s+B/s} \xrightarrow{\frac{1}{N_m \text{sec}}} \frac{1}{\text{sec}} \]

OR:

\[ V \xrightarrow{A/s+\alpha} \]

Adding to this a tachometer and motor driver, we get

\[ X(s) \xrightarrow{K_D} V \xrightarrow{A/s+\alpha} W_m = Y(s) \]

Now apply the PI control to this block diagram.

The above "tutorial" was by necessity short. I hope this will help you in your lab speed control projects.

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