Transient Response of a Second-Order System

ECEN 2830

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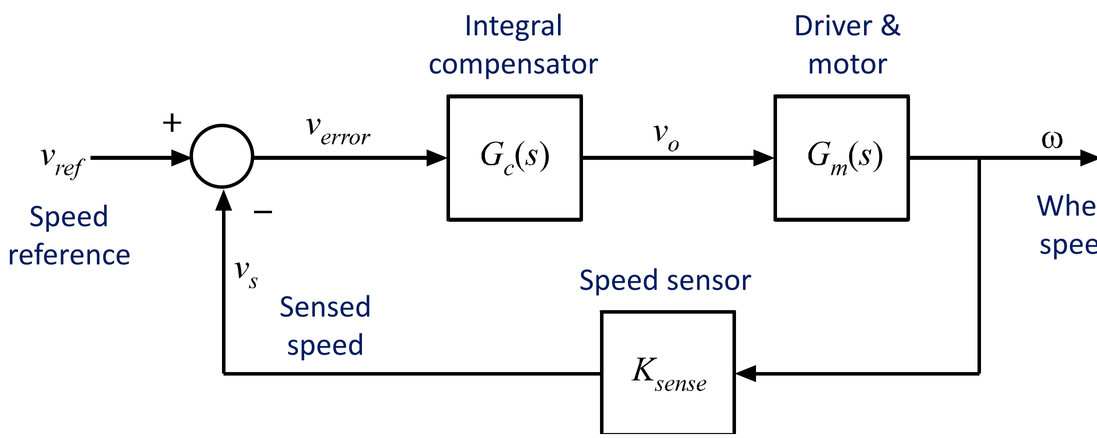
1. Introduction

In connection with this experiment, you are selecting the gains in your feedback loop to obtain a well-behaved closed-loop response (from the reference voltage to the shaft speed). The transfer function of this response contains two poles, which can be real or complex. This document derives the step response of the general second-order step response in detail, using partial fraction expansion as necessary.



2. Transient response of the general second-order system

Consider a circuit having the following second-order transfer function $H(s)$:

$$\frac{v_{out}(s)}{v_{in}(s)} = H(s) = \frac{H_0}{1 + 2\zeta \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2} \tag{1}$$

where H_0, ζ, and ω_0 are constants that depend on the circuit element values K, R, C, etc. (For our experiment, v_{in} is the speed reference voltage v_{ref}, and v_{out} is the wheel speed ω) In the case of a passive circuit containing real positive inductor, capacitor, and resistor values, the parameters ζ and ω_0 are positive real numbers. The constants H_0, ζ, and ω_0 are found by comparing Eq. (1) with the actual transfer function of the circuit. It is common practice to measure the transient response of the circuit using a unit step function $u(t)$ as an input test signal:

$$v_{in}(t) = (1\text{ V})\, u(t) \tag{2}$$

The initial conditions in the circuit are set to zero, and the output voltage waveform is measured.

This test approximates the conditions of transients often encountered in actual operation. It is usually desired that the output voltage waveform be an accurate reproduction of the input (i.e., also a step function). However, the observed output voltage waveform of the second order system deviates from a step function because it exhibits ringing, overshoot, and nonzero rise time. Hence, we might try to select the component values such that the ringing, overshoot, and rise time are minimized.

The output voltage waveform $v_{out}(t)$ can be found using the Laplace transform. The transform of the input voltage is

$$v_{in}(s) = \frac{1}{s} \tag{3}$$

The Laplace transform of the output voltage is equal to the input $v_{in}(s)$ multiplied by the transfer function $H(s)$:

$$v_{out}(s) = H(s)\, v_{in}(s) = \frac{1}{s} \, \frac{H_0}{1 + 2\zeta \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2} \tag{4}$$

The inverse transform is found via partial fraction expansion.

The roots of the denominator of $v_{out}(s)$ occur at $s = 0$ and (by use of the quadratic formula) at

$$s_1, s_2 = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1} \tag{5}$$

Three cases occur:

- $\zeta > 1$. The roots s_1 and s_2 are real. This is called the *overdamped* case.
- $\zeta = 1$. The roots s_1 and s_2 are real and repeated: $s_1 = s_2 = -\zeta\omega_0$. This case is called *critically damped*.
- $\zeta < 1$. The roots s_1 and s_2 are complex, and can be written

  $$s_1, s_2 = -\zeta\omega_0 \pm j\omega_0\sqrt{1 - \zeta^2} \tag{6}$$

  This is called the *underdamped* case.

Figure 2 illustrates how the positions of the roots, or poles, vary with ζ. For $\zeta = \infty$, there are real poles at $s = 0$ and at $s = -\infty$. As ζ decreases from ∞ to 1, these real poles move towards each other until, at $\zeta = 1$, they both occur at $s = -\omega_0$. Further decreasing ζ causes the poles to become complex conjugates as given by Eq. (6). Figure 3 illustrates how the poles then move around a circle of radius ω_0 until, at $\zeta = 0$, the poles have zero real parts and lie on the imaginary axis. Figure 2 is called a *root locus* diagram, because it illustrates how the roots of the denominator polynomial of $H(s)$ move in the complex plane as the parameter ζ is varied between 0 and ∞.

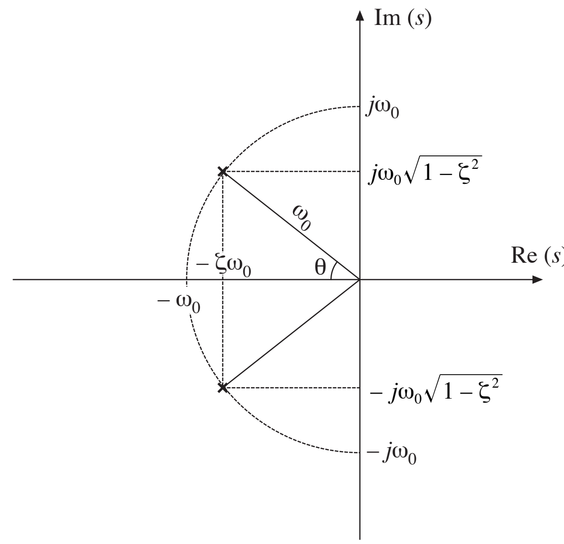


Fig. 3. For $0 \le \zeta < 1$, the complex conjugate poles lie on a circle of radius ω_0.

Several other cases can be defined that are normally not useful in practical engineering systems. When $\zeta = 0$, the roots have zero real parts. This is called the *undamped* case, and the output voltage waveform is sinusoidal. The transient excited by the step input does not decay for large t. When $\zeta < 0$, the roots have positive real parts and lie in the right half of the complex plane. The output voltage response in this case is *unstable*, because the expression for $v_{out}(t)$ contains exponentially growing terms that increase without bound for large t.

Partial fraction expansion is used below to derive the output voltage waveforms for the cases that are have useful engineering applications, e.g. the overdamped, critically damped, and underdamped cases.

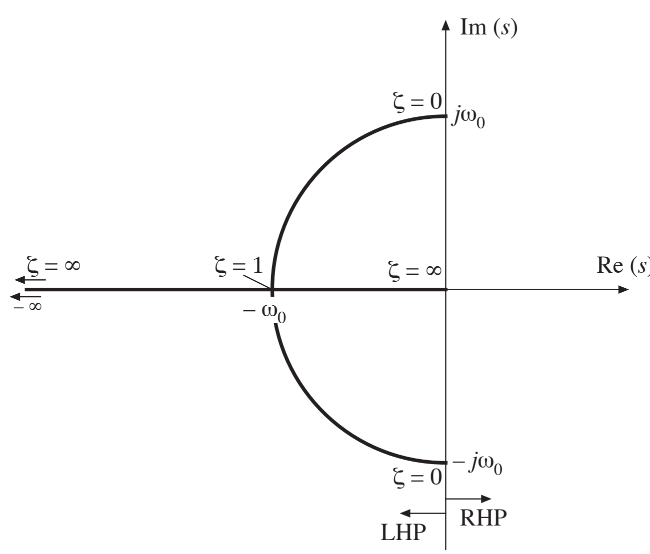


Fig. 2. Location of the two poles of $H(s)$ vs. ζ, as described by Eqs. (5) and (6).

2.1. Overdamped case, $\zeta > 1$

Partial fraction expansion of Eq. (4) leads to

$$v_{out}(s) = \frac{K_1}{s} + \frac{K_2}{s - s_1} + \frac{K_3}{s - s_2} \tag{7}$$

Here, s_1 and s_2 are given by Eq. (5), and the residues K_1, K_2 and K_3 are given by

$$K_1 = s\left(\frac{1}{s}\,\frac{H_0}{1 + 2\zeta\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}\right)\Bigg|_{s=0}$$

$$K_2 = \left(s - s_1\right)\left(\frac{1}{s}\,\frac{H_0}{1 + 2\zeta\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}\right)\Bigg|_{s=s_1}$$

$$K_3 = \left(s - s_2\right)\left(\frac{1}{s}\,\frac{H_0}{1 + 2\zeta\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}\right)\Bigg|_{s=s_2} \tag{8}$$

Evaluation of these expressions leads to

$$K_1 = H_0$$

$$K_2 = \left(s - s_1\right)\left(\frac{1}{s}\,\frac{H_0}{\left(1 - \frac{s}{s_1}\right)\left(1 - \frac{s}{s_2}\right)}\right)\Bigg|_{s=s_1} = -\frac{s_2}{s_2 - s_1}H_0$$

$$K_3 = \left(s - s_2\right)\left(\frac{1}{s}\,\frac{H_0}{\left(1 - \frac{s}{s_1}\right)\left(1 - \frac{s}{s_2}\right)}\right)\Bigg|_{s=s_2} = -\frac{s_1}{s_1 - s_2}H_0 \tag{9}$$

The inverse transform is therefore

$$v_{out}(t) = H_0 u(t)\left[1 - \frac{s_2}{s_2 - s_1}e^{s_1 t} - \frac{s_1}{s_1 - s_2}e^{s_2 t}\right] \tag{10}$$

In the overdamped case, the output voltage response contains decaying exponential terms, and the rise time depends on the magnitudes of the roots s_1 and s_2. The root having the smallest magnitude dominates Eq. (10): for $|s_1| << |s_2|$, Eq. (10) is approximately equal to

$$v_{out}(t) \approx H_0 u(t)\left[1 - e^{s_1 t}\right] \tag{11}$$

This is indeed what happens when $\zeta >> 1$. Equation (11) can be expressed in terms of ω_0 and ζ as

$$v_{out}(t) \approx H_0 u(t)\left[1 - e^{-\frac{\omega_0 t}{2\zeta}}\right] \tag{12}$$

When $\zeta >> 1$, the time constant $2\zeta/\omega_0$ is large and the response becomes quite slow.

2.2. Critically damped case, $\zeta = 1$

In this case, Eq. (4) reduces to

$$v_{out}(s) = \frac{H_0}{s\left(1 + \frac{s}{\omega_0}\right)^2} \tag{13}$$

The partial fraction expansion of this equation is of the form

$$v_{out}(s) = \frac{K_1}{s} + \frac{K_2}{\left(s + \omega_0\right)^2} + \frac{K_3}{\left(s + \omega_0\right)} \tag{14}$$

with the residues given by

$$\begin{aligned} K_1 &= H_0 \\ K_2 &= \left.\left(s + \omega_0\right)^2 \left(\frac{H_0}{s\left(1 + \frac{s}{\omega_0}\right)^2}\right)\right|_{s = -\omega_0} = -\omega_0 H_0 \\ K_3 &= \left.\frac{d}{ds}\left[\left(s + \omega_0\right)^2 \left(\frac{H_0}{s\left(1 + \frac{s}{\omega_0}\right)^2}\right)\right]\right|_{s = -\omega_0} = -H_0 \end{aligned} \tag{15}$$

The inverse transform is therefore

$$v_{out}(t) = H_0 u(t)\left[1 - \left(1 + \omega_0 t\right) e^{-\omega_0 t}\right] \tag{16}$$

In the critically damped case, the time constant $1/\omega_0$ is smaller than the slower time constant $2\zeta/\omega_0$ of the overdamped case. In consequence, the response is faster. This is the fastest response that contains no overshoot and ringing.

2.3. Underdamped case, $\zeta < 1$

The roots in this case are complex, as given by Eq. (6). The partial fraction expansion of Eq. (4) is of the form

$$v_{out}(s) = \frac{K_1}{s} + \frac{K_2}{s + \zeta\omega_0 - j\omega_0\sqrt{1 - \zeta^2}} + \frac{K_2^*}{s + \zeta\omega_0 + j\omega_0\sqrt{1 - \zeta^2}} \tag{17}$$

The residues are computed as follows:

$$K_1 = H_0$$

$$K_2 = \left(s + \zeta\omega_0 - j\omega_0\sqrt{1-\zeta^2}\right)\left(\frac{H_0\omega_0^2}{s\left(s + \zeta\omega_0 - j\omega_0\sqrt{1-\zeta^2}\right)\left(s + \zeta\omega_0 + j\omega_0\sqrt{1-\zeta^2}\right)}\right)\Bigg|_{s = -\zeta\omega_0 + j\omega_0\sqrt{1-\zeta^2}} \tag{18}$$

The expression for K_2 can be simplified as follows:

$$\begin{aligned} K_2 &= \frac{H_0\omega_0^2}{\left(-\zeta\omega_0 + j\omega_0\sqrt{1-\zeta^2}\right)\left(2j\omega_0\sqrt{1-\zeta^2}\right)} \\ &= \frac{H_0}{\left(-\zeta + j\sqrt{1-\zeta^2}\right)\left(2j\sqrt{1-\zeta^2}\right)} \\ &= -\frac{H_0}{2\left(1-\zeta^2\right) + j2\zeta\sqrt{1-\zeta^2}} \end{aligned} \tag{19}$$

The magnitude of K_2 is

$$\left\| K_2 \right\| = \frac{H_0}{2\sqrt{1-\zeta^2}} \tag{20}$$

and the phase of K_2 is

$$\begin{aligned} \angle K_2 &= \tan^{-1}\frac{2\zeta\sqrt{1-\zeta^2}}{2\left(1-\zeta^2\right)} \\ &= \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}} \end{aligned} \tag{21}$$

The inverse transform is therefore

$$v_{out}(t) = H_0 u(t)\left[1 - \frac{1}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_0 t}\cos\left(\sqrt{1-\zeta^2}\,\omega_0 t + \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right] \tag{22}$$

In the underdamped case, the output voltage rises from zero to H_0 faster than in the critically-damped and overdamped cases. Unfortunately, the output voltage then overshoots this value, and may ring for many cycles before settling down to the final steady-state value.

In some applications, a moderate amount of ringing and overshoot may be acceptable. In other applications, overshoot and ringing is completely unacceptable, and may result in destruction of some elements in the system. The engineer must use his or her judgment in deciding on the best value of ζ.

3. Step response waveforms

Equations (10), (16), and (19) were employed to plot the step response waveforms of Fig. 4. Underdamped, critically damped, and overdamped responses are shown. It can be deduced from Fig. 4 that the parameter ω_0 scales the horizontal (time) axis, while H_0 scales the vertical (output voltage) axis. The damping factor ζ determines the shape of the waveform.

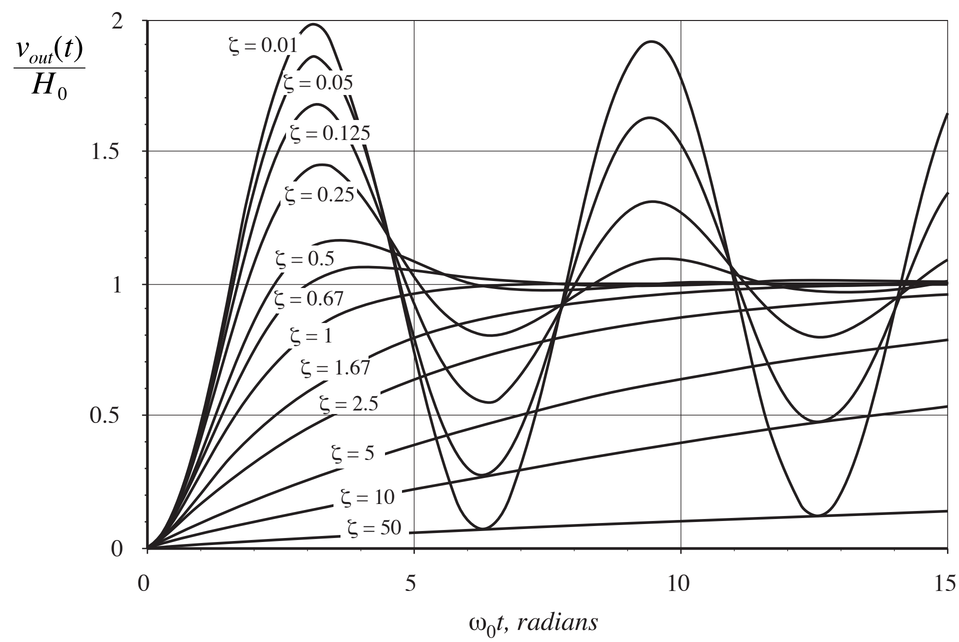


Fig. 4. Second-order system step response, for various values of damping factor ζ.

Three figures-of-merit for judging the step response are the *rise time,* the *percent overshoot,* and the *settling time*. Percent overshoot is zero for the overdamped and critically damped cases. For the underdamped case, percent overshoot is defined as

$$percent\ overshoot = \left(\frac{peak\ v_{out} - v_{out}(\infty)}{v_{out}(\infty)} \right) \left(100\% \right) \tag{20}$$

One can set the derivative of Eq. (19) to zero, to find the maximum value of $v_{out}(t)$. One can then plug the result into Eq. (20), to evaluate the percent overshoot. Note that the final (steady-state) value of the output $v_{out}(\infty)$ is H_0. The following equation for the percent overshoot results:

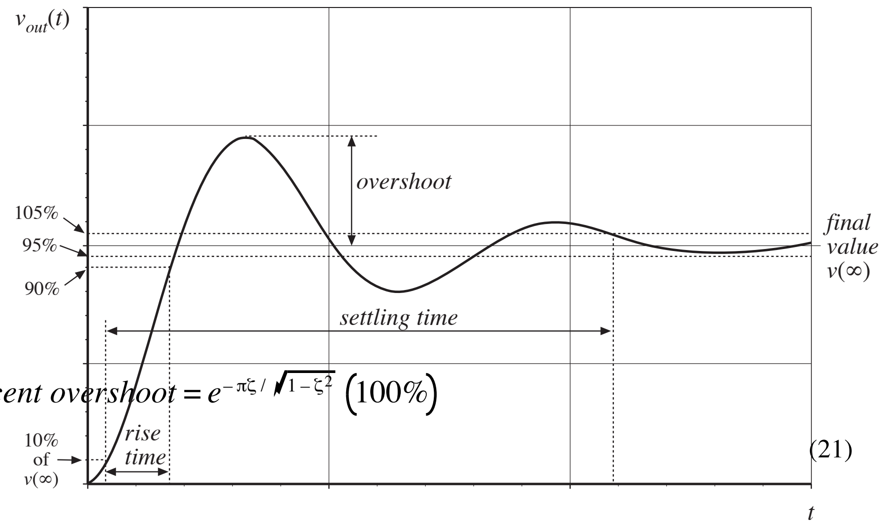


Fig. 5. Salient features of step response, second order system.

$$percent\ overshoot = e^{-\pi\zeta/\sqrt{1-\zeta^2}}\left(100\%\right) \quad (21)$$

Again, this equation is valid only in the underdamped case, i.e., for $0 < \zeta < 1$. It can be seen from Fig. 4 that decreasing the damping factor ζ results in increased overshoot. The overshoot is 0% for $\zeta = 1$. In the limit of $\zeta = 0$ (the undamped case), the overshoot approaches 100%.

As illustrated in Fig. 5, the *rise time* is defined as the time required for the output voltage to rise from 10% to 90% of its final steady-state value. When the system is underdamped, the output waveform may pass through 90% of its final value several times; the first pass is used in computation of the rise time. It can be seen from Fig. 4 that the rise time increases monotonically with increasing ζ.

The *settling time* is the time required by an underdamped system for its output voltage response to approach steady state and stay within some specified percentage (for example, 5%) of the final steady-state value. As can be seen from Fig. 4, systems having very small values of ζ have short rise times but long settling times.

4. Experimental measurement of step response.

The difficulty in measuring a transient response is that it happens only once—if you blink, you will miss it! This problem can be alleviated by causing the step input to be periodic: apply a square wave (Fig. 6) to the circuit input. The duration $T/2$ of the positive portion of the square wave is chosen to be much longer than the settling time of the output response, so that the circuit is in steady-state just before each step of the input waveform occurs. In consequence, the

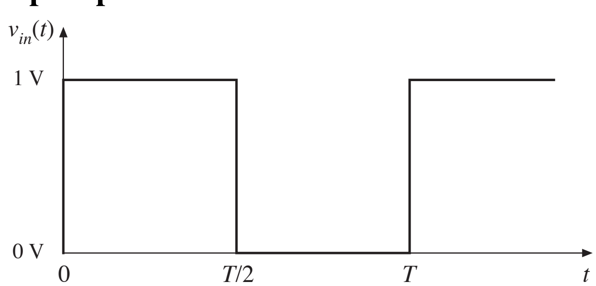


Fig. 6. Use of a square wave input, with sufficiently long period T, allows the output transient to be observed on any oscilloscope.

output voltage waveform is identical to the waveform observed when a single step input is applied, except that the output transient occurs repetitively. The output transient waveform can now be easily observed on an oscilloscope, and can be studied in detail.

BIBLIOGRAPHY

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