11

Some Applications of Electrostatics

11.1 Introduction

At this point, we will briefly study some common instances and applications of electrostatic fields. The earth’s strongest electric field is the atmospheric field between charged clouds, and between charged clouds and the ground. We will briefly describe fields, voltages, and currents for the extreme case of thunderstorms. Commercial applications of electrostatics include pollution-control filters, xerography, printing, electrostatic separation, coating, and some medical and biological applications. We will also look at some applications where understanding of dc current density is needed, such as probes for characterizing semiconductor materials. We will review only the basic principles of these processes and include some examples of engineering design parameters that directly apply the knowledge gained in the previous chapters. Even at that level, the examples in this chapter will show clearly how powerful the knowledge we have gained is for understanding existing devices and for discovering and designing new ones.
11.2 Atmospheric Electricity and Storms

Thunderstorms are the most obvious manifestations of electrical phenomena on our planet. A typical storm cloud carries about 10 to 20 coulombs of each type of charge, at an average height of 5 km above the earth’s surface. Where does the electric energy of a cloud come from? The main source of energy on our planet is the sun: huge water masses on earth are heated by the sun’s energy. The evaporated water, which contains energy originated from the sun, forms clouds. A small portion of the energy of a cloud is turned into electric energy. So thunderstorms are huge (but quite inefficient) thermoelectric generators. Some of the suspected mechanisms of cloud electrification are the breakup of raindrops, freezing, frostling, and friction between drops or crystals. Movements inside a cloud eventually make the cloud into an electric dipole, with negative charges in the lower part and positive charges in the upper part. The lower part induces positive charges on the earth below the cloud (Fig. 11.1a). The induced charge density is greatest on tall, sharp objects.

Figure 11.1 Formation of a lightning bolt. (a, b) The stepped-leader down-flowing stroke defines the path for the visible return stroke (c, d).
The beginning of a cloud-to-ground lightning bolt is an invisible discharge, called the stepped leader. The stepped-leader air breakdown is initiated at the bottom of the cloud. It moves in discrete steps, each about 50 m long and lasting for about 1 \( \mu \text{s} \). Because of this discharge, electrons are released from the lower part of the cloud. These electrons are attracted by the induced positive charges on the ground, and they move downward (Fig. 11.1b). As the negatively charged stepped leader approaches the ground, it induces even more positive charges, especially on protruding objects. When the leader is about 100 m above the ground, a spark moves up from the ground to meet it. Once a conductive path is established, huge numbers of electrons flow from the cloud to the ground. To balance the charge flow, positive charges move up toward the cloud, trying to neutralize the huge negative charge at the bottom of the cloud. This is the discharge that we see and is called the return stroke. The return stroke lasts only about 100 \( \mu \text{s} \), so it looks as if the entire channel lights up at the same time.

The currents in the return stroke are typically on the order of 10 kA but can be as large as 200 kA. The temperatures associated with the Joule’s heat of such a current are very high—temperatures in the return stroke reach 30,000 K. The air does not have time to expand in volume, so its pressure rises to several million pascals, causing a sound shock—thunder.

At any time, there are about 1800 thunderstorms across the earth, and about 100 lightning flashes per second. The National Center for Health Statistics estimates that the death toll of lightning, about 150 people every year in the United States, is bigger than that of hurricanes and tornadoes. Lightning also causes considerable damage. Lightning rods (connected to grounding electrodes) protect exposed structures from damage by routing the strokes to the ground through the rod rather than through the structure. Benjamin Franklin suggested the use of lightning rods, and they were in place in the United States and France as early as the middle of the 18th century. It is estimated that for rural buildings containing straw and protected by a rod, the danger of fire caused by lightning is reduced by a factor of 50.

Questions and problems: Q11.1 to Q11.3, P11.1 and P11.2

11.3 Electric Current in a Vacuum and in Gases

In lightning, current flows from the clouds to the earth through atmosphere, which is a gas. There are two major differences between electric currents in solid and liquid conductors and those in a vacuum and in rarefied (low-pressure) gases. First, the electric charges in a vacuum or in gases are most often moved by the electric field of some stationary charges only; there is usually no impressed electric field. Second, in a vacuum and in gases there is no relation similar to the point form of Ohm’s law. This is obvious in the case of a vacuum: charges do not collide with atoms but move under the influence of the electric forces and forces of inertia only. Consequently they do not follow the lines of vector \( \mathbf{E} \) except if the field lines are straight.

In the case of gases, particularly if rarefied, an accelerated ion has a relatively long path between collisions, so it can acquire a considerable kinetic energy. As a consequence, in rarefied gases Ohm’s law is not valid. Various other effects can oc-
cur, however, due to the possibility of a chain production of new pairs of ions by collisions of high-velocity ions with neutral molecules. Electrostatic discharge due to high voltages is one such effect that is of practical interest and will be described briefly in this chapter.

**Example 11.1—Motion of electric charges in the electric field.** Let a charge $Q$ of mass $m$ move in an electrostatic field in a vacuum. The equation of motion of the charge has the form

$$m\frac{d^2r(t)}{dt^2} = QE(r),$$

(11.1)

where $r(t)$ is the position vector (variable in time) of the charge, and $E$ is the electric field strength, a function of coordinates (that is, of $r$).

Let us now look specifically at the motion of a charged particle in a uniform electric field. Assume that a charge $Q$ ($Q < 0$) leaves with a negligibly small velocity the negative plate of a charged parallel-plate capacitor in which the electric field strength is $E$. Let the plates be a distance $d$ apart. We wish to determine the position and velocity of the charge as a function of time, if the charge left the plate at $t = 0$.

Let the $x$ axis be perpendicular to the plates, with $x = 0$ at the negative plate (Fig. 11.2). The charge will move parallel to the $x$ axis. According to Eq. (11.1), the equation of motion is

$$m\frac{d^2x}{dt^2} = QE,$$

that is, the charge moves with constant acceleration $QE/m$. The charge velocity as a function of time is given by

$$v = \frac{dx}{dt} = \frac{QE}{m}t,$$

![Figure 11.2 Motion of a charged particle in a uniform electric field between two parallel charged plates](image-url)
since the initial velocity is zero. From this equation, the position of the particle is given by

\[ x = \frac{Q E}{2m} t^2. \]

Assume that the voltage between the plates is \( V = 1000 \text{ V} \), that the charge is an electron \( (e \simeq -1.6 \cdot 10^{-19} \text{ C}, m \simeq 9.1 \cdot 10^{-31} \text{ kg}) \), and that the distance between the plates is \( d = 10 \text{ cm} \). The electric field strength between the plates is then \( E = V/d = 10,000 \text{ V/m} \).

The charge will reach the positive plate after a time obtained from the last equation, in which we set \( x = d \). This yields \( t = 1.07 \cdot 10^{-8} \text{ s} \). The velocity of the electron before impact with the positive electrode is now obtained if we insert this particular time into the expression for the velocity. The result is \( v = 18.8 \cdot 10^6 \text{ m/s} \), a very large velocity (close to those velocities for which relativistic corrections in particle mass need to be made). In conclusion, electrons can be accelerated to remarkably high velocities with voltages that are not difficult to produce.

The preceding equations were derived for a uniform electric field. For arbitrary \( \mathbf{E}(\mathbf{r}) \) no analytical solution of Eq. (11.1) for the position vector \( \mathbf{r} \) in time is known, but it can always be found approximately by numerical methods. Often the exact trajectory of the particle is not of interest. Instead only the magnitude of the velocity of the particle is required, and it can easily be calculated.

Let the charged particle (of charge \( Q \) and mass \( m \)) leave point 1, which is at potential \( V_1 \), with a velocity of magnitude \( v_1 \). We wish to determine the magnitude \( v_2 \) of its velocity when it reaches point 2, which is at a potential \( V_2 \). In moving the particle from point 1 to point 2, the electric forces perform work \( Q(V_1 - V_2) \), and due to energy conservation, we know that the particle kinetic energy must have increased by precisely that amount. Thus

\[ \frac{mv_2^2}{2} - \frac{mv_1^2}{2} = Q \int_1^2 \mathbf{E} \cdot d\mathbf{l} = Q(V_1 - V_2). \]  

(11.2)

The magnitude of the velocity at point 2 is

\[ v_2 = \sqrt{v_1^2 + \frac{2Q(V_1 - V_2)}{m}}. \]  

(11.3)

In the particular, but common, case when \( v_1 = 0 \), this becomes

\[ v_2 = \sqrt{\frac{2Q(V_1 - V_2)}{m}}. \]  

(11.4)

(Velocity of a charge accelerated from zero velocity by potential difference \( V_1 - V_2 \))

**Example 11.2—Velocity of an electron accelerated by 1 kV.** As a numerical example, let us determine the velocity of an electron accelerated from zero velocity by a 1000-V voltage. Using Eq. (11.4) we get \( v = 18.8 \cdot 10^6 \text{ m/s} \), as in Example 11.1 for the special case of a uniform field. Note that this result is valid for any electric field (not necessarily uniform), in which the electron covers a voltage of 1000 V.

*Questions and problems: Q11.4 to Q11.9, P11.3 to P11.7*
11.4 Corona and Spark Discharge

In gases, and particularly in air at normal atmospheric pressure, specific steady discharging currents may occur in certain circumstances. A necessary condition for this process is a region of electric field with intensity greater than the dielectric strength of air (about 30 kV/cm). In this region the air becomes ionized, i.e., conducting, which is equivalent to an enlargement of the electrode. If the electric field intensity outside this enlarged "electrode" is less than the dielectric strength of air, the process stops. The cloud of charges around the electrode stays permanently, and it forms a source of ions that are propelled toward the electrodes of the opposite sign. As a result, there are steady discharging currents between the electrodes. The ionized cloud is known as a corona.

In some instances, however, the electric field strength may not be decreased when a corona is formed around an electrode. The process then does not stop, but instead spans the whole region between the electrodes. Violent discharge of the electrodes occurs, known as spark discharge. The spark discharge is not, of course, a time-invariant phenomenon.

Normally corona is not desirable, because it results in losses of charge on charged conductors. In some instances, however, it is of great use. For example, discharging of an aircraft that is charged during flight by friction is performed by encouraging corona discharges at several positions on the aircraft. We will see in the next sections that corona discharge is done on purpose when small neutral metal or dielectric particles need to be charged, for example in electrostatic painting of cars.

Spark discharge is also usually undesirable. For example, in manufacturing plants or coal mines where explosive gases may exist, electrostatically charged bodies may discharge through sparks, which in turn may have enough energy to initiate a large-scale explosion. Spark discharge is a relatively frequent cause of explosions involving loss of human lives and property.

Questions and problems: Q11.10 to Q11.12

11.5 Electrostatic Pollution-Control Filters

Electrostatic filters are used in environmental control for removing fine particles from exhaust gases. In the filtering (or precipitation) process, the particles are charged, separated from the rest of the gas by a strong electric field, and finally attracted to a pollutant-collecting electrode. In the United States there are a few thousand large industrial electrostatic filters, and a large number of small units used for indoor air cleaning. Electric power generation plants in the United States generate about $2 \times 10^7$ tons of coal fly ash every year. This ash accounts for most of the use of electrostatic filters, although steel and cement production, paper processing, sulfuric acid manufacturing, petroleum refining, and phosphate and other chemical processing also use electrostatic filters.

A simplified diagram of a filter (this type is referred to as the "tubular" precipitator) is shown in Fig. 11.3. The polluted gas flow enters the bottom of the cylinder and the small ash particles are charged by ionized air around a high-potential
electrode (to learn more about this mechanism of gas ionization, the reader is encouraged to refer to, e.g., A.D. Moore, ed., *Electrostatics and its applications*, John Wiley, 1973). The electric field between the high-potential electrode and the coaxial grounded cylinder causes the charged particles to be attracted to the cylinder, which acts as the collecting electrode. The purified gas flows through the clean-gas exhaust at the top of the cylinder.

To understand the engineering design problems in electrostatic filters, let us consider first an uncharged idealized spherical conductive particle in a uniform electric field (Fig. 11.4a). Due to the presence of the external field, induced charges are created on the surface of the sphere. Using knowledge we gained in Chapter 6, it can be shown that the resulting electric field at points A and B on the sphere is three times that of the external field (see Example 11.3).

This means that the uniform external field has been changed by the presence of the uncharged particle and is now nonuniform, as shown in Fig. 11.4a. (The field inside the particle is not shown, because it is different for conductive or dielectric particles, as will be discussed shortly.) We said that in an electrostatic filter these particles encounter ionized air when they enter the electric field. This means that they are not strictly in a vacuum, since there are occasional charged ions in the space around them. These ions will be attracted to the particle if they are found in a certain region close to it, as shown in Fig. 11.4a. In this case, the particle becomes charged, and the excess charge distributes itself to satisfy the boundary conditions. This has an effect on the surrounding field, which becomes more nonuniform than in the previous case, as shown in Fig. 11.4b. The result is that it will be more difficult for the next ion of the same sign to be attracted to the particle, because the size of the region where it needs to find itself in order to be attracted is reduced.

The particles will be collected faster if they carry more charge, because then the electric force is larger. However, we have just explained that particles cannot be
charged beyond a certain amount. From this relatively simple example, we can see that there is a limitation on how fast pollutant particles can be charged and collected inside an electrostatic filter. In order to speed up the process of collection, and therefore filter out as many pollutant particles as possible in a limited region of exhaust-gas flow inside the filter, different designs than the one shown in Fig. 11.4 have been in use. More details can be found in H.J. White, *Industrial electrostatic precipitation*, Addison-Wesley, 1963.

**Example 11.3—Dielectric and conductive spherical particles in a uniform electric field.** A dust particle can be approximated by a dielectric or conductive sphere in a (locally) uniform field. Consider first a uniformly polarized sphere of radius $a$. Let the polarization vector in the sphere be $\mathbf{P}$ (Fig. 11.5a). We assume for the moment that the sphere is situated in a vacuum, and that its polarization is the only source of the field.

The field outside and inside the sphere is the same as the field resulting only from the surface polarization charges. (Since $\mathbf{P}$ is constant, $\rho = \text{div}\mathbf{P} = 0$.) The density of these surface charges is given by Eq. (7.16), which in the case considered becomes

$$\sigma_{p} = \mathbf{P} \cdot \mathbf{n} = P \cos \theta.$$  \hspace{1cm} (11.5)
Figure 11.5 (a) A uniformly polarized dielectric sphere and (b) an uncharged conductive sphere in a uniform electric field

The field and potential of this charge distribution are not easy to find directly. However, we can use the following reasoning: the uniformly polarized dielectric sphere is equivalent to two spherical charged clouds of uniform volume densities $\rho$ and $-\rho$ whose centers are displaced by a small distance $d$. We already know that the field and potential outside such a system are equal to those of a dipole. This gives us the field and potential outside the dielectric sphere. The potential of a dipole is given by

$$V(r, \theta) = \frac{p \cos \theta}{4\pi \varepsilon_0 r^2},$$

so all we need to find is the equivalent dipole moment $p$ for the two spherical clouds. Recall the definition of the polarization vector:

$$\mathbf{P} = \sum \frac{d\mathbf{p}}{dV}.$$

Since all the $\mathbf{p}$ moments in $dV$ are parallel, and $d\mathbf{p} = (\rho \, dV) \mathbf{d}$, we get

$$P = \rho d = \frac{Q}{4a^2\pi/3} d = \frac{p}{4a^2\pi/3}.$$

So the dipole moment of the two displaced charged spheres is

$$p = \frac{4}{3}a^3\pi P,$$

and thus we know the potential (and hence also the field) outside the uniformly polarized sphere.

We wish now to determine the potential (and the field) inside the sphere. Consider the potential

$$V(r, \theta) = \frac{pr \cos \theta}{4\pi \varepsilon_0 a^3} = \frac{px}{4\pi \varepsilon_0 a^3} \quad (r \leq a).$$

(11.7)

It is a simple matter to prove that this potential satisfies Laplace’s equation (so it is a physically possible potential). For $r = a$, it becomes identical with the potential outside the sphere. Since the potential is continuous across the sphere surface, we conclude that boundary conditions
are satisfied, and that the expression in Eq. (11.7) represents the potential inside the uniformly polarized sphere.

The electric field inside the dielectric sphere has only an \( x \) component, and is given by

\[
E_x = \frac{\partial V}{\partial x} = -\frac{p}{4\pi \epsilon_0 a^3} = -\frac{p}{3\epsilon_0}.
\]

Thus, the electric field inside the uniformly polarized sphere is uniform, and in the \( x \) direction.

Consider now a dielectric sphere in a uniform external electric field \( E_0 \mathbf{u}_x \). This field, of course, tends to uniformly polarize all dielectric bodies in it. However, polarization charges for irregular bodies will produce an irregular secondary field, and generally the polarization of the bodies will not be uniform. Only if the shape of the body is such that a uniform polarization of the body results in a uniform secondary electric field inside it will the polarization of the body in the end also be uniform. We have just demonstrated that for a dielectric sphere this is precisely the case. Thus, inside a dielectric sphere in a uniform field the total field is uniform.

To determine the polarization of the sphere and then its secondary field using the preceding equations, note that \( \mathbf{P} = (\epsilon - \epsilon_0) \mathbf{E} \). In this case, \( \mathbf{E} \) is the total electric field inside the sphere, equal to the sum of the external field, \( E_0 \mathbf{u}_x \), and the field in Eq. (11.8). It is left to the reader as an exercise to determine the polarization of the sphere, and hence the total field inside and outside the sphere.

In some cases, pollutant particles are conductive rather than insulating. For example, a dust particle can be approximated by a conductive sphere, as shown in Fig. 11.5b. When such an uncharged metal sphere is placed in a uniform electric field \( \mathbf{E} \), charges are induced on its surface to cancel the electric field inside the sphere. Since the external field is uniform, the charges have to distribute themselves to produce an opposite uniform field inside the sphere. From the previous discussion of a dielectric sphere in a uniform electric field, we know that this charge distribution has to be of the form in Eq. (11.5). The electric field strength in Eq. (11.8) is \( E \). Consequently, the induced surface charge on the sphere is of density

\[
\sigma(\theta) = 3\epsilon_0 E \cos \theta.
\]

Since \( D_n = \epsilon_0 E_n = \sigma \), the largest value of the field on the surface of the ball is for \( \theta = 0 \) and \( \theta = \pi \) (points \( A \) and \( B \) in the figure):

\[
E_A = E_B = 3E.
\]

At points \( A \) and \( B \) on the surface of a conducting dust particle, the electric field is three times stronger than the original uniform field.

Questions and problems: Q11.13 and Q11.14, P11.8

11.6 Electrostatic Imaging—Xerography

The modern photocopy machine was invented by the physicist and lawyer Chester Carlson in 1938. In his patent work he saw the need for an inexpensive and easy way to copy documents. It took him about 10 years to develop the copier, and in 1947 the Haloid Company—now Xerox Corporation—licensed the invention and began commercial production. The first copier was introduced to the market in 1950. Carlson called the process xerography from the Greek words xeros "dry," and graphos,
"writing." Xerography uses a photosensitive material called selenium. Selenium is normally a dielectric, but when illuminated it becomes conductive. Some other materials, such as zinc oxide and anthracene, also have this property.

The copying process essentially has five steps, shown in Fig. 11.6. In the first step, a selenium-coated plate is charged evenly by sliding it under positively charged wires. An image of the document is then exposed onto the plate by a camera lens. In places where the plate is illuminated (corresponding to the white areas of the document), the selenium becomes a conductor and the charge flows away to metal contacts on the side of the plate. In other places, corresponding to the dark (printed) areas

Figure 11.6 Five essential steps in the xerography process: (1) charging of the photoconductor plate; (2) charge image formation; (3) development with negatively charged toner; (4) transfer of toner image to paper; and (5) image fixing by heating
of the document, the charge remains. The plate now has an exact copy of the original in the form of a positive charge pattern. In the next step, some toner is charged negatively and the toner particles are attracted to the positively charged copy and stick to the plate. A sheet of blank paper is then placed over the plate and powder toner image. The paper is positively charged, so it attracts the toner particles onto itself. The paper is then placed on a fuser tray and the toner is baked to seal the image permanently. The entire process in the first copier took about 3 minutes. Modern copiers are faster and more sophisticated, but they operate according to the same principles.

What are some important electrostatic engineering design parameters in copiers? We said that the illuminated parts of the selenium plate become conductive and the charge flows away. But can the charge in the remaining charge image stay in place indefinitely, or do we need to time the next step in the copying process according to the temporal stability of the charge image? To answer this question, we need to know what the electric resistivity ($\rho$) and permittivity ($\varepsilon$) of dark selenium (selenium with no illumination) are.

Once these two properties are known, we can reason in the following way. At the surface of the film, Gauss' law gives

$$\oint_{S_0} \mathbf{E} \cdot d\mathbf{S}_0 = \frac{1}{\varepsilon} \int_S \sigma \mathbf{n} \cdot d\mathbf{S}, \quad (11.11)$$

where $\sigma$ is the surface charge density of the image on the film, $S_0$ is a thin, coinlike closed surface, with one base inside the film and the other over its very surface, and $S$ is the intersecting surface of $S_0$ and the film. The continuity equation can also be written for the current density $\mathbf{J} = \mathbf{E}/\rho$ flowing through the film due to its finite resistivity $\rho$:

$$\oint_{S_0} \mathbf{E} \cdot d\mathbf{S}_0 = -\rho \int_S \frac{\partial \sigma}{\partial t} \mathbf{n} \cdot d\mathbf{S}. \quad (11.12)$$

Eqs. (11.11) and (11.12) must be valid for any shape of the intersecting surface $S_0$. This is possible only if the expressions on their right sides are equal. In that case, the two equations can be combined to give a differential equation for the surface charge density, $\sigma$:

$$\frac{d\sigma}{dt} + \frac{1}{\varepsilon \rho} \sigma = 0. \quad (11.13)$$

Assuming that at $t = 0$ the surface charge density is $\sigma_0$, the solution of this equation is $\sigma = \sigma_0 e^{-1/(\varepsilon \rho)}$. The quantity $\varepsilon \rho$ describes how quickly the charge image on the film diffuses. This quantity is called the charge transfer time constant, or dielectric relaxation constant of dark selenium. It tells us how long it takes for $\sigma_0/e \simeq 0.368 \sigma_0$ of the charge in the image to flow away ($e = 2.7182 \ldots$ is the base of natural logarithms). For selenium, the resistivity varies between $10^{11} \Omega \cdot m$ and $10^{14} \Omega \cdot m$, and the relative permittivity is about 6.1. The corresponding charge transfer time constants are 5.4 s to 5400 s (1.5 hours). This means that in the former case, after the charge image is formed by illumination, the toner image needs to be formed and transferred to paper in less than a few seconds in order to create a clear image.
Note that the charge transfer time constant $\epsilon \rho$ is in some sense similar to the RC time constant of a resistor-capacitor circuit. We have seen before that the capacitance $C$ of a dielectric-filled capacitor is proportional to $\epsilon$, and the resistance $R$ is proportional to the resistivity $\rho$.

Another practical problem in copiers is developing, i.e., making a toner image from the charge image. The toner consists of small dielectric particles, about 10 $\mu$m in diameter, which are charged to about $Q = 0.5 \cdot 10^{-14}$ C. These particles are brought close to the photoconductive film, where a strong electric field, only a few times weaker than the air breakdown field, exists wherever there is a charge image. The electric force on a toner particle for a field strength three times smaller than the breakdown of air is then about

$$F = QE = 0.5 \cdot 10^{-14} \text{C} \cdot 10^6 \text{V/m} = 0.5 \cdot 10^{-8} \text{N}. \quad (11.14)$$

If this force were the only one present, the toner particles would move exactly along the electric field vector lines. However, as we discussed in Example 11.3, if we assume that each particle is a tiny sphere, it becomes a dipole, and there is a force in addition to the force on the toner particles we just calculated, due to the inhomogeneous electric field of the charge image. After expressing the polarization vector $\mathbf{P}$ in Example 11.3 in terms of the permittivity of the dielectric, it can be shown that this additional force due to the field nonuniformity is about three orders of magnitude smaller than the force calculated in Eq. (11.14), and can therefore be neglected. The toner particles also have mass, and therefore a gravitational force is acting on them. This force is given by $F_g = 4\pi r^3 \rho g / 3 \simeq 0.5 \cdot 10^{-11}$ N for a spherical particle 5 $\mu$m in radius and with mass density equal to that of water ($\rho = 10^3 \text{kg/m}^3$). So the gravitational force can also be neglected, and our original conclusion that the toner particles follow the selenium-film electric field lines is a good approximation.

How is the toner brought to the charge image? Of several possible processes the following one is probably the simplest. A fine dust of carbon particles (on the order of $\bullet \mu$m) is electrified and blown over the charge image, covering the charged parts of the image. If the paper surface has appropriate properties, the image is transferred onto the paper efficiently. Note that the small size of the dust particles enables a very high resolution image.

One of the engineering problems in this process is the fact that the electric field distribution around the edges of the image is nonuniform, as shown in Fig. 11.7a.

![Figure 11.7 Field distribution around the edges of a charge image](image-url)

(a) without and (b) with the presence of a developing conductive plate
The field is the strongest around the edges, and toner particles are attracted only to the areas around the edges. In order to obtain toner coverage in the areas inside the image edges, a grounded conducting surface (called a developing electrode) is brought very near the charged film, but not touching it. The electric field strength between the photoconductive surface and the electrode is proportional to the surface charge density. The toner is introduced between the two surfaces, gets attracted to the surface charge on the film, and neutralizes this charge. The quantity of toner needed to effectively neutralize the charge at a certain spot on the film is proportional to the original surface charge density, which in turn is inversely proportional to the intensity of illumination during optical exposure. Thus the density of the developed image reproduces the continuous tones of the original optical image.

Questions and problems: Q11.15 to Q11.18, P11.9

11.7 Industrial Electrostatic Separation

An important application of electrostatic fields is separation, used in industry for purification of food, purification of ores, sorting of reusable wastes, and sizing (sorting according to size and weight). Some specific examples of electrostatic separation in the ore and mineral industry are as follows: quartz from phosphates; diamonds from silica; gold and titanium from beach sand; limestone, molybdenite, and iron ore (hematite) from silicates; and zircon from beach sand. In the food industry, peanut beans, cocoa beans, walnuts, and nut meats are separated electrostatically from shells. For grains, electrostatics is used to separate rodent excrement from barley, soybean, and rice. In the electronics industry, copper wire is electrostatically separated from its insulation for recycling purposes. It is estimated that well over 10 million tons of products a year are processed using electrostatic separation.

The first patent in this field was issued in 1880 for a ground cereal purification process. Thomas Edison had a patent in 1892 for electrostatically concentrating gold ore, and the first commercial process used in a plant in Wisconsin in 1908 was set up to electrostatically purify zinc and lead ores. Shortly after that, flotation processes were invented for separation. However, as these processes are not suitable for arid areas, and in some cases they require chemical reagents that present water pollution problems, electrostatic separation has regained popularity. In 1965, the world’s largest electrical concentration plant was installed in the Wabush Mines in Canada. This plant is used to reduce the silica content of 6 million tons of iron ore per year.

Figure 11.8 shows two basic systems of electrostatic separation (there are several others not described here). In each case, the basic components of the system are a charging mechanism, an external electric field, a device to regulate the trajectory of nonelectric particles, and a feeding and collection system. In case (a) the particles are charged by contact electrification (the triboelectric effect), and in case (b) by ion bombardment (corona discharge). Both of these effects were mentioned earlier in this chapter. In terms of regulating the trajectory of the particles, in case (a) the gravitational force is used in addition to the electric force, and in case (b) the centrifugal force acting on the particles is adjusted by a rotating cylinder.
In each case, physical separation of two types of particles is performed by adjusting the forces acting on the particles, as well as the time the forces act, so that different types of particles will have different trajectories. In Example 11.4, an approximate analysis of the separation process is examined for case (a) in Fig. 11.8. In case (b), used for separation of hematite from quartz or copper wire from its insulation, the particles are charged through ion bombardment. One type of particle is conductive, and the other dielectric. The particles are dropped on top of a grounded rotating cylinder. The conductor particles share their charge with the grounded rotor and are thrown from the rotor in a trajectory determined by centrifugal forces, gravity, and air resistance. The dielectric particles are held to the surface of the rotor. They either lose their charge slowly and then fall off the rotor, or are scraped off by a brush at the other end. The electric and centrifugal forces need to be roughly the same in order for the particles to stick to the rotor long enough for good separation.

**Example 11.4—Separation of quartz from phosphate rock using gravitational and electric forces.** An approximate analysis of the forming chute separation of quartz from phosphate rock can be done by ignoring the electric forces acting on particles due to neighboring charges. Obviously, this analysis is very rough and provides us only with an order-of-magnitude illustration.

The quartz and phosphate rock particles are washed, dried, heated to about 100°C, and finally vibrated so that the minerals make and break contact and are charged by friction with charges of opposite sign. For a uniform electric field acting along the $x$ direction in Fig. 11.8a, Newton’s first law gives us an equality between the electric force on a single particle and its mass multiplied by its acceleration,

$$ F_x = QE = m \frac{d^2x}{dt^2} u_x. $$

(11.15)
which can be integrated. Assuming zero initial velocity and displacement, the expression for the deflection of a charged particle due to the electric force is

\[ x = \frac{1}{2} \frac{QE}{m} t^2. \] (11.16)

For a 0.25-mm-diameter quartz particle, the ratio \( Q/m \approx 9 \times 10^{-6} \text{ C/kg} \), and a typical value for the electric field strength is \( E = 4 \times 10^5 \text{ V/m} \). Therefore, Eq. (11.16) gives \( x = 1.8t^2 \text{ m} \). The time required for a particle to fall a certain distance is obtained from the expression for the gravitational force

\[ F_g = mg = m \frac{d^2 y}{dt^2} u_y. \] (11.17)

The vertical displacement of the particle due to the gravitational force is found by integration to be

\[ y = -\frac{1}{2} gt^2. \] (11.18)

For a falling distance of, say, 0.5 m, we can now calculate the time that the electric force has to deflect the particle in the horizontal direction as \( t^2 \approx 0.1 \text{s}^2 \), and the horizontal displacement as \( d \approx 18 \text{ cm} \). In the case of two particles that are oppositely charged, after falling 0.5 m they are separated by 36 cm, which is enough for good separation.

The engineering limitations of this process relate to the fact that the process can be used only for a certain range of particle sizes: if the particles become too large, roughly larger than 1 mm in diameter, the gravitational force becomes too large compared to the electric force. If the particles are too small, below roughly 50 \( \mu \text{m} \), interparticle attractive electric forces cause the small quartz and phosphate particles to form clusters.

**Questions and problems:** Q11.19, P11.10

### 11.8 Four-Point Probe for Resistivity Measurements

Four-point probes are used in every semiconductor lab. They can be used for determining the resistivity (conductivity) of a material, or the charge concentration if the other material properties are known. First consider just two probes (two points) that are touching the surface of a material of unknown conductivity, as in Fig. 11.9. The idea behind using probes is the same as measuring the resistance of a resistor, except that some of the quantities are distributed (i.e., described at every point, not only at the two resistor terminals).

When a current source is connected to the two probes in Fig. 11.9, the current density in the material can be found using the image theorem described in Chapter 10 when we discussed grounding electrodes. The current density vector is the superposition of the current density from the point from which the current is “injected” into the material and the current density from the point out of which the current is returning to the generator (which is just the negative of the first current with respect
to the surface of the material). As shown in the figure, this current density vector can easily be found only along the line connecting the two probes right underneath the material-air interface. Once we know the current density $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$, we can use Ohm's law $\mathbf{E} = \rho \mathbf{J}$ to find the expression for the voltage drop between the two probes in terms of the source current $I$ and the unknown resistivity $\rho$:

$$V = \int_{\text{probe 1}}^{\text{probe 2}} \mathbf{E} \cdot d\mathbf{l} = \int_{\text{probe 1}}^{\text{probe 2}} \rho \mathbf{J} \cdot d\mathbf{l}.$$ 

It is left as an exercise for the reader to write the expression for the current density and to attempt solving this integral. When solving the integral, note that the limits of integration should be the radii of the probe contacts. But because it is almost impossible to accurately know the radii of the probe contacts, we cannot accurately measure the properties of a solid material by this method.

To avoid this difficulty in measuring resistivity we use a four-point probe instead (Fig. 11.10). This time, the current is injected into the material from the outer
two probes, but the voltage is sampled between the two inner probes. Because the two voltage probes can be pointlike, and because they are far from the current probes of radii known only approximately, this results in a much more accurate measurement of the resistivity of the material. It is left as an exercise (P11.11) to find the expression for the resistivity in this case.

Questions and problems: Q11.20, P11.11 to P11.17

11.9 Brief Overview of Other Applications

Many other applications also involve electrostatic fields, but in this section we will briefly describe only some of them.

One of the most common electrostatic applications is coating. In the car industry several coats of paint are applied to vehicles using electrostatic coating. Every washer, dryer, and refrigerator is electrostatically coated. Even such objects as golf balls and the paper you now hold in front of you have been coated for some purpose (the paper is coated in order to have a good printing surface). The basic principle of electrostatic coating is simple: the object to be coated is charged with one polarity, and the coating material with another. The coating material is sprayed into fine particles around the object and the particles are attracted to the object by electric forces and deposited upon impact. Of course, coating machines in industry are quite sophisticated because it is important to achieve uniform coats and also to use the coating material efficiently.

Imaging technology also uses an electrostatic-based device, the charge-coupled device (CCD) camera. CCD design can even be used, for example, in making the extremely sensitive cameras used in astronomy. A CCD camera consists of a large array of MOS capacitors (Fig. 11.11). We discussed MOS capacitors in Example 8.9. Incident light creates both positive and negative charge carriers inside the p semiconductor (silicon). The metal electrodes are biased positively, so that the electrons are attracted to the semiconductor-oxide interface. The number of electrons under each metal electrode is an accurate measure of the number of incident light photons. How does the camera reconstruct the image as something we can see? It needs, in effect,
to measure the number of electrons in each capacitor. This is done by moving each group of electrons along a line in a serial fashion after this electron "pulsed" current is amplified in an amplifier at the end of the line. In the readout direction, each of the electrodes is biased to a progressively higher voltage, so that the electrons from the neighboring electrode are attracted to it. This transfer process can be very efficient, and in good devices only 1 out of 100,000 electrons is lost in each charge transfer step.

Another commercial application of electrostatics is nonimpact printing, for example in ink-jet printers. A basic diagram of an ink-jet printer appears in Fig. 11.12. The ink jet, on the order of 100 μm in diameter, is produced by applying pressure to an ink supply. The droplet stream, initially uncharged, passes through a cylindrical charging electrode biased at around 100 V along its axis, so that the jet and the electrode form a coaxial capacitor. The ink supply is connected to the other generator terminal, so the jet is charged by electrostatic induction. The jet is next modulated mechanically so that it turns into charged droplets, between 25 and 125 μm in diameter for 0.254-cm-high characters. The charged droplets are then deflected into the desired dot-matrix pattern by the electrostatic field between two metal plates with a voltage of 1 to 5 kV between them. The deflection in one plane is controlled by the voltage between the electrodes. The other dimension is usually controlled by the mechanical motion of the stream with respect to the paper.

Other applications of electrostatics include electrostatic motors, electrostatic generators, and electrophoresis (separation of charged colloidal particles by the electric field) used in biology (for example, to separate live yeast cells from dead ones). Electrophoresis is used in biochemistry to separate large charged molecules by placing them in an electric field. For example, in genetic research, DNA molecules of different topological forms (e.g., supercoiled and linear DNA) are separated effectively by electrophoresis, even though they are chemically identical. Hewlett-Packard has also developed an electrophoresis instrument intended for use in the drug industry. The interested reader is referred to the Hewlett-Packard Journal, June 1995.

Questions and problems:  Q11.21
QUESTIONS

Q11.1. Describe the formation of a lightning stroke.
Q11.2. How large are currents in a lightning stroke?
Q11.3. According to which physical law does thunder occur?
Q11.4. A spherical cloud of positive charges is allowed to disperse under the influence of its own repulsive forces. Will charges follow the lines of the electric field strength vector?
Q11.5. A cloud of identical, charged particles is situated in a vacuum in the gravitational field of the earth. Is there an impressed electric field in addition to the electric field of the charges themselves? Explain.
Q11.6. Is Eq. (11.1) valid if the charge Q from time to time collides with another particle?
Q11.7. Explain why the left-hand side in Eq. (11.2) is as it is, and not \( n(v_2 - v_1)^2/2 \).
Q11.8. Discuss the validity of Eqs. (11.3) and (11.4) if the charge Q is negative.
Q11.9. An electron is emitted parallel to a large conducting flat plate that is uncharged. Describe qualitatively the motion of the electron.
Q11.10. If the voltage between the electrodes of an air-filled parallel-plate capacitor is increased so that corona starts on the plates, what will eventually happen without increasing the voltage further?
Q11.11. Electric charge is continually brought on the inner surface of an isolated hollow metal sphere situated in air. Explain what will happen outside the sphere.
Q11.12. Give a few examples of desirable and undesirable (1) corona and (2) spark discharges.
Q11.13. Describe how an electrostatic pollution-control filter works.
Q11.14. Sketch the field that results when an uncharged spherical conductive particle is brought into an originally uniform electric field.
Q11.15. Describe the process of making a xerographic copy.
Q11.16. Explain the physical meaning of the charge transfer time constant, or dielectric relaxation constant.
Q11.17. Derive Eq. (11.13) and solve it.
Q11.18. Describe the difference in the xerographic image with and without the developer plate.
Q11.19. Derive the equation of particle trajectory in a forming chute process for electrostatic separation.
Q11.20. Why is a four-point probe measurement more precise than a two-point probe measurement?
Q11.21. Describe how a CCD camera works.

PROBLEMS

P11.1. Calculate the voltage between the two feet of a person (0.5 m apart), standing \( r = 20 \text{ m} \) away from a 10-kA lightning stroke, if the moderately wet homogeneous soil conductivity is \( 10^{-2} \text{ S/m} \). Do the calculation for the two cases when the person is standing in positions A and B as shown in Fig. P11.1.
P11.2. Calculate the electric field strength above a tree that is \( d = 1 \) km away from the projection of the center of a cloud onto the earth (Fig. P11.2). Assume that because the tree is like a sharp point, the field above the tree is about 100 times that on the flat ground. As earlier, you can assume the cloud is an electric dipole above a perfectly conducting earth, with dimensions as shown in the figure, and with \( Q = 4 \) C of charge. (Note that the height of any tree is much smaller than the indicated height of the cloud.)

P11.3. Derive the second equation in Example 11.1 from Eq. (11.1).

P11.4. Assuming that the initial velocity in Example 11.1 is nonzero and \( x \)-directed, solve for the velocity and the position of the charge \( Q \) as a function of time. Plot your results.

P11.5. Assuming that the initial velocity in Example 11.1 is nonzero and \( y \)-directed, solve for the velocity and the position of the charge \( Q \) as a function of time. Plot your results.

P11.6. A thin electron beam is formed with some convenient electrode system. The electrons in the beam are accelerated by a voltage \( V_0 \). The beam passes between two parallel plates, which electrostatically deflect the beam, and later falls on the screen \( S \) (Fig. P11.6). Determine and plot the deflection \( y_0 \) of the beam as a function of the
voltage \( V \) between the plates. (This method is used for electrostatic deflection of the electron beam in some cathode-ray tubes.)

**P11.7.** A beam of charged particles that have positive charge \( Q \), mass \( m \), and different velocities enters between two closely spaced curved metal plates. The distance \( d \) between the plates is much smaller than the radius \( R \) of their curvature (Fig. P11.7). Determine the velocity \( v_0 \) of the particles that are deflected by the electric field between the plates so that they leave the plates without hitting any of them. Note that this is a kind of filter for charged particles, resulting in a beam of particles of the same velocity.

**P11.8.** A metal sphere is placed in a uniform electric field \( E_0 \). What is the maximum value of this field that does not produce air breakdown when the metal ball is brought into it?

**P11.9.** Calculate the dielectric relaxation constants for selenium, \( n \)-doped silicon with carrier concentration \( n = 10^{16} \text{ cm}^{-3} \), and \( n \)-doped gallium arsenide with concentration \( n = 10^{16} \text{ cm}^{-3} \). For semiconductors, such as silicon and gallium arsenide, the conductivity is given by \( \sigma = Q\mu n \), where \( Q \) is the electron charge. \( \mu \) is a property of electrons inside a material, and it is called the mobility (defined as \( v = \mu E \), where \( v \) is the velocity of charges that moved by a field \( E \)). For silicon, \( \mu = 0.135 \text{ m}^2/\text{Vs} \) and \( \eta_r = 12 \), and for gallium arsenide, \( \mu = -0.86 \text{ m}^2/\text{Vs} \) and \( \epsilon_r = 11 \). for selenium, \( \rho = 10^{12} \Omega\text{-m} \) and \( \epsilon_r = 6.1 \).
P11.10. How far do 1-mm-diameter quartz particles charged with \( Q = 1 \) pC need to fall in a field \( E = 2 \times 10^5 \) V/m in order to be separated by 0.5 m in a forming chute separation process? The mass density of quartz is \( \rho_m = 2.2 \) g/cm\(^3\).

P11.11. Find the expression for determining resistivity from a four-point probe measurement, as in Fig. P11.11.

![Figure P11.11 A four-point probe measurement](image)

P11.12. Using the information given in P11.19, for a measured resistivity of 10 \( \Omega \)-cm, determine the corresponding charge concentration of (1) silicon and (2) gallium arsenide.

P11.13. A Wenner array used in geology is shown in Fig. P11.13. This instrument is used for determining approximately the depth of a water layer under ground. First the electrodes are placed close together, and the resistivity of soil is determined. Then the electrodes are moved farther and farther apart, until the resistivity measurement changes due to the effect of the water layer. Assuming that the top layer of soil has a very different conductivity than the water layer, what is the approximate spacing between the probes, \( r \), that detects a water layer at depth \( h \) under the surface? The exact analysis is complicated, so think of an approximate qualitative solution.

![Figure P11.13 A Wenner array used in geology](image)
**P11.14.** A thin film of resistive material is deposited on a perfect insulator. Using a four-point probe measurement, determine the expression for surface resistivity \( \rho_s \) of the thin film. Assume the film is very thin.

**P11.15.** Consider an approximate circuit equivalent of a thin resistive film as in Fig. P11.15. The mesh is infinite, and all resistors are equal and have a value of \( R = 1 \, \Omega \). Using a two-point probe analogy, determine the resistance between any two adjacent nodes \( A \) and \( B \) in the mesh.

![Figure P11.15 An approximate equivalent circuit of a thin resistive film](image)

**P11.16.** Find the resistance between nodes (1) \( A \) and \( C \) and (2) \( A \) and \( D \) in Fig. P11.15.

**P11.17.** Construct an approximate equivalent circuit for a block of homogeneous resistive material. Determine the resistance between two adjacent nodes of the equivalent circuit.

*Problem P11.16 is not solvable using elementary symmetry techniques. The general solution is in Zemanian (1991)*