23

Waveguides and Resonators

23.1 Introduction

Waveguides are structures that direct electromagnetic energy along a desired path, transmission lines being just one example. We know that transmission lines consist of two conductors, but some may have more than two, as in three-phase power lines. Maxwell’s equations predict that electromagnetic waves can also be guided through hollow metallic tubes, like water is “guided” through pipes. There are a variety of such hollow metallic waveguides, differing in the shape of their cross section; the most common shape is rectangular.

Maxwell’s equations also predict that electromagnetic waves can be guided by dielectric slabs or rods, known as dielectric waveguides. For example, an optical fiber is a specific type of dielectric waveguide used for guiding electromagnetic waves at optical frequencies.

Transmission lines can support waves with vectors \( E \) and \( H \) in planes \textit{transversal} (normal) to the direction of wave propagation. We know that such waves are termed \textit{transverse electromagnetic waves}, or TEM waves. We already know that plane waves are also TEM waves, but for a plane wave the vectors \( E \) and \( H \) in transversal planes are \textit{constant}, whereas in transmission lines they are not.

Waveguides in the form of metallic tubes and dielectric plates or rods cannot support TEM waves. Instead, waves along such waveguides may have \textit{either} the \( E \) vector \textit{or} the \( H \) vector in the transversal plane alone, but the other vector must have...
a component in the direction of propagation. These two wave types are called transverse electric, or TE, waves, and transverse magnetic, or TM, waves.

We know that lossless transmission lines guide TEM waves of any frequency, and with the same velocity. We will see that TE and TM waves can propagate only above a certain critical frequency, and that their velocity depends on frequency. So structures supporting TE and TM waves behave as high-pass filters.

We have seen that among other purposes, transmission lines are used as circuit elements (to obtain an element with desired reactance, to act as a transformer, etc.). Waveguides are also used for such purposes, but only in the microwave range because they would be very large and impractical at low frequencies. They are used as building blocks of various microwave components: attenuators, phase shifters, transformers, and so on. We will consider only one such component, the so-called resonant cavity, which is an analogue to an LC resonant circuit with a lumped inductor and a lumped capacitor. A resonant cavity, however, is a spatial resonator, in the form of a box in which electromagnetic energy oscillates, similar to the way acoustic energy oscillates in a hallway.

The theory of waveguides is significantly more complex than any theory we have considered so far, and a complete presentation is beyond the scope of this introductory text. Since the waveguides are of great practical importance at higher frequencies, every electrical engineer should know at least the basic concepts of these electromagnetic structures. A compromise is therefore made in what follows, and most of the basic waveguide equations are given without proof. The interested reader can find these proofs in Appendix 8.

### 23.2 Wave Types (Modes)

Consider a hollow, perfectly conducting waveguide pipe, filled with a perfect dielectric of parameters $\epsilon$ and $\mu$, as in Fig. 23.1. Let the complex vectors $E$ and $H$ in the waveguide be of the form $E_{\text{tot}} = E(x, y)e^{-\gamma z}$ and $H_{\text{tot}} = H(x, y)e^{-\gamma z}$. Here $\gamma$ is the propagation coefficient in the $z$ direction. First we allow $\gamma$ to be complex, and later we will discuss what that physically means. After performing vector differentiation

![Figure 23.1 Cross section of a general waveguide. It is assumed that the waveguide is lossless, and that the dielectric is homogeneous.](image)
on Maxwell's equations in complex form, as shown in detail in Appendix 8, the following expressions for the electric and magnetic field components are obtained:

\[ E_x = -\frac{1}{K^2} \left( \gamma \frac{\partial E_z}{\partial x} + j \omega \mu \frac{\partial H_z}{\partial y} \right), \quad (23.1) \]

\[ E_y = -\frac{1}{K^2} \left( \gamma \frac{\partial E_z}{\partial y} - j \omega \mu \frac{\partial H_z}{\partial x} \right), \quad (23.2) \]

\[ H_x = -\frac{1}{K^2} \left( -j \omega \epsilon \frac{\partial E_z}{\partial y} + \gamma \frac{\partial H_z}{\partial x} \right), \quad (23.3) \]

\[ H_y = -\frac{1}{K^2} \left( j \omega \epsilon \frac{\partial E_z}{\partial x} + \gamma \frac{\partial H_z}{\partial y} \right), \quad (23.4) \]

where

\[ K^2 = \gamma^2 + \beta^2, \quad \beta^2 = \omega^2 \epsilon \mu. \quad (23.5) \]

That Eqs. (23.1) to (23.5) are solutions to Maxwell's equations can be checked by substitution. Note that the propagation coefficient \( \gamma \) (and therefore also the coefficient \( K \)) is not known. So there are seven scalar unknowns (the six field components and \( \gamma \)).

We can reach an interesting conclusion by looking carefully at the preceding equations: if we can find \( E_2(x, y) \) and \( H_2(x, y) \), we know the complete electromagnetic field for a given waveguide shape and size. [Thus the functions \( E_2(x, y) \) and \( H_2(x, y) \) play a role analogous to a potential function because all the other components are obtained from them by differentiation.]

These equations have three classes of solution:

1. Both \( E_z = 0 \) and \( H_z = 0 \), that is, only transversal components of the wave exist. [The possibility of such a solution is not evident from Eqs. (23.1) to (23.5), but will be demonstrated in the next section.] This solution corresponds to a TEM wave.

2. \( E_z = 0 \), but \( H_z \neq 0 \). This solution corresponds to a transverse electric (TE) wave.

3. \( E_z \neq 0 \), and \( H_z = 0 \). This solution corresponds to a transverse magnetic (TM) wave.

We now examine these three classes of solutions, often called modes, in turn.

23.2.1 TRANSVERSE ELECTROMAGNETIC (TEM) WAVES

The salient properties of TEM wave types, or TEM modes, are the TEM propagation coefficient \( \gamma \), the wave impedance \( Z_{TEM} \), and the unique quasi-static nature of TEM wave types.

**Propagation Coefficient**

If both \( E_z = 0 \) and \( H_z = 0 \), the expressions in parentheses in Eqs. (23.1) to (23.4) are zero. One might be tempted to conclude that all the other components are also zero.
Note, however, that $K$ is not known, so it can also be zero. We know that the expression of the form $0/0$ need not be undefined (for example, $\sin x/x \rightarrow 1$ if $x \rightarrow 0$). So the solution could exist only if $K^2 = \gamma^2 + \beta^2 = 0$, or

$$\gamma = \pm j\omega \sqrt{\epsilon \mu}. \quad (23.6)$$

(Propagation coefficient of TEM waves)

We recognize in $\gamma$ the propagation coefficient of plane waves, and also the propagation coefficient along lossless transmission lines. We will see that indeed, waves propagating along lossless transmission lines are of the TEM type.

**Wave Impedance**

In Eqs. (23.1) and (23.4) let $\gamma = \pm j\omega \sqrt{\epsilon \mu}$. After simple manipulations we find that in such a case, the ratio of the transverse electric and magnetic field components is

$$\frac{E_x}{H_y} = \pm Z_{\text{TEM}}, \quad \frac{E_y}{H_x} = \mp Z_{\text{TEM}}, \quad \text{where} \quad Z_{\text{TEM}} = \sqrt{\frac{\mu}{\epsilon}} \quad (23.7)$$

(Wave impedance of TEM waves)

for any $E_z$ and $H_z$ (which cancel out). $Z_{\text{TEM}}$ is known as the wave impedance of TEM waves.

From this we can draw three conclusions. First, vector $H$ is normal to vector $E$ (both are, of course, in a transverse plane). Second, the ratio of the electric and magnetic fields for a forward wave (the upper sign) is the intrinsic impedance of the medium, and for the backward wave it is the negative of that. Third, for the forward and for the backward waves $E$ and $H$ are such that their cross product results in the Poynting vector (power flow) in the respective direction. How do these properties compare to those of a plane wave in free space?

**Quasi-Static Nature of TEM Waves**

There is an interesting general conclusion about TEM wave types. It turns out (see Appendix 8, section A8.2) that the electric field is derivable from a potential function which at $z = 0$ satisfies Laplace’s equation in $x$ and $y$:

$$\frac{\partial^2 V(x, y)}{\partial x^2} + \frac{\partial^2 V(x, y)}{\partial y^2} = 0. \quad (23.8)$$

Because boundary conditions require that the tangential $E$ on conductor surfaces be zero, we reach the following conclusion: for TEM waves, the electric field in planes where $z$ is constant is the same as the electrostatic field corresponding to the potentials of waveguide conductors at that cross section.

**Example 23.1**—A TEM wave cannot propagate through a hollow metal tube. Consider a waveguide in the form of a hollow metal tube. For a TEM wave to exist inside the tube, the field must be the same as in the electrostatic case. But we know that inside a hollow conductor
with no charges there can be no electrostatic field. Consequently, TEM waves cannot propagate through hollow metallic waveguides.

Note that a coaxial cable does have another conductor in the tube, and that an electrostatic field can exist in the cable if the two cable conductors are at different potentials. Therefore, a TEM wave can propagate inside a coaxial cable (which we already know is true).

**Example 23.2**—Transmission lines must have at least two conductors. For the electrostatic field to exist in a cylindrical system, we must have at least two conductors. Indeed, a single charged conductor is a fiction—it implies infinite electrical energy per unit length, since the potential of the conductor with respect to a reference point at infinity is infinite. So TEM waves cannot propagate along a single wire.

However, any electrostatic system of two or more conductors with a zero total charge per unit length is feasible, because it has a finite electrical energy per unit length. Consequently, TEM waves can propagate along such waveguides. Note that this also implies a zero total current at any cross section of a transmission line, a proof of which is left as an exercise for the reader. Thus, equations of TEM waves propagating along waveguides are actually equations of wave propagation along lossless transmission lines.

### 23.2.2 TRANSVERSE ELECTRIC (TE) WAVES

Now let us briefly examine the general properties of TE wave types (for details, see Appendix 8, section A8.3).

**Propagation Coefficient**

Using the condition \( E_z(x, y) = 0 \), the wave equation for the \( H_z \) component in this case is given by

\[
\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \gamma^2 H_z + \omega^2 \epsilon \mu H_z = \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + K^2 H_z = 0. \tag{23.9}
\]

To solve this equation we need to know the geometry of the waveguide. We will see that for specific boundary conditions this equation can be satisfied only for certain distinct values of the parameter \( K \). These values of \( K \), for which both the wave (Helmholtz) equation and boundary conditions are satisfied, are known as its eigenvalues, or characteristic values. We will see that, for example, eigenvalues of \( K \) for a rectangular waveguide are given by a double infinite set of pairs of numbers dependent on the waveguide dimensions and frequency. An eigenvalue of \( K \) determines the propagation coefficient \( \gamma \) according to Eq. (23.5).

**Wave Impedance**

From Eqs. (23.1) to (23.4) it follows that if \( E_z = 0 \), the transverse electric and magnetic field vectors in a TE wave are normal to each other, and that

\[
Z_{\text{TE}} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{j\omega \mu}{\gamma} \tag{23.10}
\]

(Wave impedance of TE waves)
is a constant, the same at all points of the field in a waveguide. This is known as the 
wave impedance of TE modes. We will see that it is not the same as that for TEM waves, 
because $\gamma$ for TE waves is different from $j\omega\sqrt{\varepsilon\mu}$.

23.2.3 TRANSVERSE MAGNETIC (TM) WAVES

Finally, let us look at the general properties of TM wave types.

Propagation Coefficient

Using the condition $H_z(x, y) = 0$, we can obtain $E_z$ from the Helmholtz equation, 
which now reads

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + K^2 E_z = 0. \quad (23.11)$$

To solve this equation we need to know the geometry of the waveguide, and a solu­
tion exists only for specific values of $K$ (its eigenvalues), as in the case of TE modes.

Wave Impedance

As in the case of TE modes, from Eqs. (23.1) to (23.4) it follows that for $H_z = 0$, the 
transverse electric and magnetic field vectors in a TM wave are normal to each other, 
and that

$$\frac{E_x}{H_y} = -\frac{E_y}{H_x} = Z_{TM} = \frac{\gamma}{j\omega\varepsilon} \quad (23.12)$$

(Wave impedance of TM waves)

is the same at all points. This is known as the wave impedance of TM wave types.

Note that for a forward wave and the same value of the propagation coeffi­
cient $\gamma$,

$$Z_{TE}Z_{TM} = Z_{TEM}^2 = \frac{\mu}{\varepsilon}. \quad (23.13)$$

Example 23.3—Power transmitted along waveguides. Let us now derive a general ex­
pression for the power transmitted along a waveguide. Let only a forward wave exist in a 
waveguide. The power transmitted along the waveguide can be determined by integrating the 
complex Poynting vector over the structure cross section at $z = 0$:

$$P = \int_{\Gamma_{transv}} \text{Re}\{(E_{transv} \times H_{transv}^*) \cdot u_z\} dS_{transv}, \quad (23.14)$$

where the subscript "transv" relates to components normal to the direction of propagation.

The transverse components of vectors $E$ and $H$ for all three wavetypes (TEM, TE, and 
TM) are normal to each other. In addition, their ratio equals the wave impedance of the wave,
and the cross product $\mathbf{E}_{\text{transv}} \times \mathbf{H}_{\text{transv}}^*$ is in the direction of vector $\mathbf{u}_z$. So

$$(\mathbf{E}_{\text{transv}} \times \mathbf{H}_{\text{transv}}^*) \cdot \mathbf{u}_z = E_{\text{transv}} H_{\text{transv}}^* = Z_{\text{wave \ type}} |H_{\text{transv}}|^2 = \frac{1}{Z_{\text{wave \ type}}} |E_{\text{transv}}|^2,$$

where $Z_{\text{wave \ type}}$ is the wave impedance of the wave propagating along the waveguide. Thus for the power transmitted along the waveguide we obtain

$$P_{\text{wave \ type}} = Z_{\text{wave \ type}} \int_{S_{\text{transv}}} |H_{\text{transv}}|^2 dS_{\text{transv}}$$

$$= \frac{1}{Z_{\text{wave \ type}}} \int_{S_{\text{transv}}} |E_{\text{transv}}|^2 dS_{\text{transv}} \quad \text{(valid for forward wave),} \quad (23.15)$$

where “wave type” stands for TEM, TE, or TM.

Questions and problems: Q23.1 to Q23.7, P23.1 to P23.4

### 23.3 Rectangular Metallic Waveguides

At frequencies above about 1 GHz, losses in transmission-line conductors due to skin effect become pronounced, so lower-loss hollow waveguides are used for guiding waves up to frequencies of several hundred gigahertz. Most often such waveguides are of rectangular cross section, but circular and some other cross sections are also used. We restrict our attention to rectangular waveguides.

A sketch of a rectangular waveguide is shown in Fig. 23.2. We assume the waveguide to be lossless and straight. From Example 23.1 we know that TEM waves cannot propagate along hollow waveguides. So we consider TE modes in more detail, and also TM wave types briefly. Let us start with the TE wave types.

![Figure 23.2 Sketch of a waveguide of rectangular cross section](image-url)
23.3.1 TE WAVES IN RECTANGULAR WAVEGUIDES

The complete derivation for TE modes in a rectangular metallic waveguide is given in Appendix 8, section A8.3. At the introductory level, it suffices to quote the most important expressions and discuss their practical meaning.

Complete Expression for $T_{Emn}$ Wave Types

After applying the boundary conditions to the perfectly conducting waveguide walls at $x = 0, x = a, y = 0,$ and $y = b$, the $H_z$ component in the cross section $z = 0$ of the waveguide is found to be

$$H_z(x, y) = H_0 \cos \left( \frac{m\pi}{a} x \right) \cos \left( \frac{n\pi}{b} y \right) \quad (at \ z = 0),$$

(23.16)

where $H_0$ is a constant depending on the level of excitation of the wave in the waveguide. The other components at $z = 0$ are (note that $E_z = 0$)

$$E_x(x, y) = \frac{j\omega \mu}{K^2} \frac{m\pi}{a} H_0 \cos \left( \frac{m\pi}{a} x \right) \sin \left( \frac{n\pi}{b} y \right) \quad (at \ z = 0),$$

(23.17)

$$E_y(x, y) = -\frac{j\omega \mu}{K^2} \frac{m\pi}{a} H_0 \sin \left( \frac{m\pi}{a} x \right) \cos \left( \frac{n\pi}{b} y \right) \quad (at \ z = 0),$$

(23.18)

$$H_x(x, y) = \frac{\gamma}{K^2} \frac{m\pi}{a} H_0 \sin \left( \frac{m\pi}{a} x \right) \cos \left( \frac{n\pi}{b} y \right) \quad (at \ z = 0),$$

(23.19)

$$H_y(x, y) = \frac{\gamma}{K^2} \frac{n\pi}{b} H_0 \cos \left( \frac{m\pi}{a} x \right) \sin \left( \frac{n\pi}{b} y \right) \quad (at \ z = 0).$$

(23.20)

Since the cosine and sine functions are periodic, we see that there is a double infinite number of TE wave types, corresponding to any possible pair of $m$ and $n$. Note that $m$ represents the number of half-waves along the $x$ axis, and $n$ the number of half-waves along the $y$ axis. The wave determined by $m$ and $n$ is known as a $T_{Emn}$ mode. From Eqs. (23.16) to (23.20) we see that for $m = n = 0$ all components are zero. Thus, a $T_{E00}$ mode does not exist. Waves for any other combinations of numbers $m$ and $n$ may exist, for example, $TE_{10}, TE_{01}, TE_{11}, TE_{21}$. The values of the wave components for any $z$ are obtained by simply multiplying the preceding expressions by $e^{-\gamma z} = e^{-j\beta z}$, where $\gamma$ and $\beta$ are as given in the next section.

Propagation Coefficient of TE Waves

Noting that $\omega = 2\pi f$, the expression for the propagation coefficient of a TE wave along a rectangular waveguide is

$$\gamma = j\beta \quad \beta = \omega \sqrt{\varepsilon \mu} \sqrt{1 - \left( \frac{f_c}{f} \right)^2},$$

(23.21)

(Propagation and phase coefficients of rectangular waveguides)

where $f_c$ is known as the cutoff frequency, and it depends on the mode numbers $m$ and $n$, and on the dimensions of the waveguide, $a$ and $b$. 
Cutoff Frequency of TE Wave Types

Noting that $1/\sqrt{\epsilon \mu} = c$,

$$f_c = \frac{c}{2} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}, \quad c = \frac{1}{\sqrt{\epsilon \mu}}.$$  \hspace{1cm} (23.22)

(Cutoff frequency of rectangular waveguides)

Why $f_c$ is known as the cutoff frequency is explained next.

Phase and Group Velocity of TE$_{mn}$ Waves

Let us now investigate more closely the properties of TE$_{mn}$ modes for different pairs of values of $m$ and $n$. First, note that the phase velocity of the TE$_{mn}$ mode is given by

$$v_{ph} = \frac{\omega}{\beta} = \frac{c}{\sqrt{1 - f_c^2/f^2}}.$$  \hspace{1cm} (23.23)

(Phase velocity of waves propagating along rectangular waveguides)

You may recall Example 21.8, in which we showed that for this dependence of the phase velocity on frequency, the group velocity is given by

$$v_g = c \sqrt{1 - f_c^2/f^2}.$$  \hspace{1cm} (23.24)

(Group velocity of waves propagating along rectangular waveguides)

Since the phase (and group) velocity depend on frequency, rectangular waveguides are dispersive structures. The dependence of the phase and group velocity on normalized frequency, $f/f_c$, is sketched in Fig. 23.3.

![Figure 23.3 Depedence of phase velocity and group velocity on normalized frequency, $f/f_c$.](image-url)
Assume that for a given waveguide \( f > f_c \). Then the phase coefficient is real, as are the phase velocity and the group velocity. This means that the wave of this frequency propagates along the waveguide.

If, however, \( f < f_c \), the expression under the square root in Eqs. (23.21), (23.23), and (23.24) is negative. The phase coefficient becomes imaginary, \(-\beta z\) (negative value of the root is taken to avoid exponentially increasing wave amplitudes with increasing \( z \)), so that the propagation factor \( e^{-\beta z} \) becomes \( e^{-|\beta|z} \). This means that waves of frequencies lower than \( f_c \) cannot propagate along rectangular waveguides. This is why \( f_c \) is termed the "cutoff frequency." Because the attenuation of the wave is exponential, the wave of a frequency \( f < f_c \) is attenuated very rapidly with \( z \). Thus, as mentioned in the introduction to this chapter, rectangular waveguides behave as high-pass filters.

Modes that propagate through a given waveguide are the propagating modes, and those that do not propagate are the evanescent modes. To transmit energy, we use propagating modes. Evanescent modes are used, for example, when making an attenuator out of a section of a waveguide.

Rectangular waveguides are always made such that \( a > b \). Let \( a = 2b \), which is fairly standard. The first few cutoff frequencies of the TE\(_{mn}\) modes, Eq. (23.22), relative to the cutoff frequency of the TE\(_{10}\) mode are shown in Fig. 23.4. Note that between the cutoff frequency of the TE\(_{10}\) mode and the next one, that of the TE\(_{01}\) mode, only the TE\(_{10}\) mode can propagate. This is a remarkable property of the TE\(_{10}\) mode. A discontinuity in the waveguide, like a bend or a slot in the guide wall, will always produce a multitude of modes. If we use a frequency of a wave to be within these limits (shown shaded in Fig. 23.4), out of all of these modes only the TE\(_{10}\) mode will propagate further—all other modes will be evanescent.

### 23.3.2 TM WAVES IN RECTANGULAR WAVEGUIDES

As in the case of TE modes, there are an infinite number of TM\(_{mn}\) modes, corresponding to all possible pairs of numbers \( m \) and \( n \). The expressions for the cutoff frequency, propagation coefficient, phase velocity, etc., are very similar to the TE case. There is an important difference, however: the lowest TM mode is TM\(_{11}\), that is, there is no TM mode for which either \( m = 0 \) or \( n = 0 \). As an illustration, Fig. 23.5b shows the comparison of the lowest order TE mode fields and the TM\(_{11}\) mode fields.
23.4 TE\textsubscript{10} Mode in Rectangular Waveguides

We have seen that the cutoff frequencies of all TE modes higher than the 10 mode, as well as the cutoff frequencies of all TM modes, are higher than that of the TE\textsubscript{10} mode. For this reason the TE\textsubscript{10} mode is known as the dominant mode in rectangular waveguides. It is by far the most commonly used wave type in hollow metallic waveguides, so we consider it in more detail.

The field components of the TE\textsubscript{10} mode are given by Eqs. (23.16) to (23.20) for \( m = 1 \) and \( n = 0 \):

\[
\begin{align*}
H_z(x, y) &= H_0 \cos \left( \frac{\pi x}{a} \right) \quad \text{(TE\textsubscript{10} mode)}, \\
E_x(x, y) &= E_z(x, y) = H_y(x, y) = 0 \quad \text{(TE\textsubscript{10} mode)}, \\
E_y(x, y) &= -j\omega \mu \frac{a}{\pi} H_0 \sin \left( \frac{\pi x}{a} \right) \quad \text{(TE\textsubscript{10} mode)}, \\
H_x(x, y) &= j\beta \frac{a}{\pi} H_0 \sin \left( \frac{\pi x}{a} \right) \quad \text{(TE\textsubscript{10} mode)}.
\end{align*}
\]

(Field components of TE\textsubscript{10} mode in a rectangular waveguide)

The cutoff frequency of the TE\textsubscript{10} mode is

\[
f_{c, \text{TE10}} = \frac{c}{2a},
\]

(Cutoff frequency of TE\textsubscript{10} mode in rectangular waveguide)

and the phase velocity, wave impedance, and so on for the TE\textsubscript{10} mode are obtained from the general expressions with this cutoff frequency.

The wave impedance of the TE\textsubscript{10} mode is obtained from Eqs. (23.10), (23.21), and (23.29):

\[
Z_{\text{TE10}} = \frac{\sqrt{\mu/\epsilon}}{\sqrt{1 - f_c^2/f^2}} > \sqrt{\mu/\epsilon},
\]

(23.30)

where \( f_c = c/2a \). This expression tells us that the field inside the waveguide is different from that in free space (a plane wave). Consequently, if we cut a waveguide, only part of the energy will be radiated from its open end, and the rest will be reflected back.

Example 23.4—Wavelength along waveguide. The wavelength inside the waveguide, along the z axis, is determined simply as \( \lambda_z = 2\pi/\beta \), where \( \beta \) is the phase coefficient of the mode of interest. So the wavelength along the waveguide is

\[
\frac{\lambda_z}{\lambda_0} = \frac{2\pi}{\beta} = \frac{c}{f\sqrt{1 - f_c^2/f^2}} = \frac{\lambda_0}{\sqrt{1 - f_c^2/f^2}},
\]

(Wavelength along a rectangular waveguide)

where \( \lambda_0 \) is the wavelength of a plane wave of the same frequency and in the same medium.
Figure 23.5 (a) Sketch of the E-field and H-field lines of the TE$_{10}$ mode in a rectangular waveguide, frozen in time. (b) Sketch of the E and H lines of the TM$_{11}$ mode (the mode with the lowest cutoff frequency of all TM modes). Solid lines show the E lines, and dashed lines and $\bigcirc$ and $\otimes$ symbols show the H lines. The entire picture moves at the phase velocity in the +z direction.

Example 23.5—Sketch of the field of the TE$_{10}$ mode. When we multiply Eqs. (23.25) to (23.28) by $e^{-j\omega t}$, we obtain the phasor wave components at any point $(x, y, z)$. To obtain a picture of the fields in the waveguide at an instant, we need to obtain the time-domain expressions, fix an instant in time (e.g., $t = 0$), and then plot the field lines. Although time-domain expressions are easy to obtain (this is left as an exercise for the reader), plotting the field is not simple. A sketch of the fields of the TE$_{10}$ mode is shown in Fig. 23.5a. In time, the entire picture moves in the +z direction with the phase velocity of the TE$_{10}$ mode. For comparison, the E and H lines of the TM$_{11}$ mode (the mode with the lowest cutoff frequency of all TM modes) are sketched in Fig. 23.5b.

Example 23.6—Surface current distribution on waveguide walls for the TE$_{10}$ mode. The surface currents are obtained from the time-domain expressions of the magnetic field on waveguide walls and the boundary condition in Eq. (19.12) for a perfect conductor. We need to
Figure 23.6 Sketch of surface current distribution for the TE$_{10}$ mode in a rectangular waveguide, frozen in time. The entire picture moves at the phase velocity in the $+z$ direction.

Fix an instant of time (e.g., $t = 0$), and then plot the lines of the surface-current density vector. This, again, is not an easy task. A sketch of the lines of the current density vector is shown in Fig. 23.6. In time, the surface current density distribution moves with the phase velocity of the TE$_{10}$ mode in the $+z$ direction.

Note that if we cut a slot in the waveguide wall, in such a way that the slot is always tangential to the lines of the surface current, only a small disturbance of the wave propagation in the waveguide will result. Therefore we can cut narrow slots of types $A$ and $B$ indicated in the figure without changing the fields in the waveguide. The slot of type $A$ is made to obtain a slotted waveguide used for measurements similar to those done by a slotted coaxial line (Example 18.10).

**Example 23.7—Excitation of TE$_{10}$ mode.** How can we produce a TE$_{10}$ mode? First, we need to close one end of the waveguide, to prevent propagation in both directions. We do this with a metal plate perpendicular to the waveguide, as in Fig. 23.7. In waveguide terminology, this is known as a *shorted waveguide*.

To excite the TE$_{10}$ mode, we can excite either the $E$ field or the $H$ field. The $E$ field can be excited by a small coaxial probe. A short extension of the inner conductor of a coaxial line is inserted into the waveguide, and the outer conductor of the line is connected to the waveguide wall, as in Fig. 23.7. Roughly, the position of the probe should be in the middle of the wave-
guide (where the \( E \) field is the strongest, Fig. 23.5), and about a quarter of a guided wavelength \( \lambda_z \) from the short circuit. If the latter condition is fulfilled, the wave from the probe needs one quarter of a period to reach the short circuit, is reflected there and changes phase (which is equivalent to losing another two quarters of a period), and then loses another quarter of a period to go back to the probe. So the wave reflected from the short circuit will be in phase with the wave radiated in the \( +z \) direction. This simple reasoning, however, is only a rough estimate, and the actual position of the probe needs to be determined either experimentally or using accurate numerical methods.

Another possibility is to excite the \( \text{TE}_{10} \) mode by a small loop intended to excite the \( H \) field at the place where it is the strongest, e.g., in the middle of the waveguide short circuit. It is left as an exercise for the reader to sketch this type of waveguide excitation.

Example 23.8—\( \text{TE}_{10} \) mode in X-band waveguide. The microwave frequency range is divided into so-called bands, and a waveguide of certain dimensions can support waves at frequencies covering the entire band. A commonly used range is the X band (about 8.2 to 12.4 GHz), for which a standard waveguide has \( a = 23 \) mm and \( b = 10 \) mm. Using the formulas for \( \text{TE}_{10} \) modes with these waveguide dimensions, we find that the cutoff frequency is \( f_c = 6.52 \) GHz. The guided wavelength and impedance are different at different frequencies inside the band. At the center of the band, \( f = 10 \) GHz, the guided wavelength \( \lambda_z = 3.96 \) cm, and the impedance \( Z_{\text{TE}_{10}} = 497 \) \( \Omega \). So, the characteristic impedance is much larger than that of a coaxial cable, which is usually 50 \( \Omega \). That means that the probe described in Example 23.7 needs to match the coaxial impedance to that of the waveguide dominant mode.

Example 23.9—Power transmitted by the \( \text{TE}_{10} \) mode. The power transmitted by a forward wave through any waveguide is given in Eq. (23.15). We know the transverse components and the wave impedance of the \( \text{TE}_{10} \) mode, so we need only to substitute these expressions into Eq. (23.15) and to integrate over the cross-sectional area of the waveguide,

\[
P_{\text{TE}_{10}} = \frac{\beta}{\omega \mu} \int_{y=0}^{b} \int_{x=0}^{a} \frac{\omega \mu \mu_0^2 \frac{a^2}{\pi^2} |H_0|^2 \sin^2 \left( \frac{\pi x}{a} \right) dx dy = \frac{\omega \mu \beta a^2 |H_0|^2}{\pi^2} \frac{a}{2} \frac{a}{2} = \frac{\omega \mu \beta a^3 b |H_0|^2}{2\pi^2}.
\]

\( H_0 \) is the rms value of the magnetic field at the magnetic field maximum. If \( \beta \) is replaced by its expression in Eq. (23.21), this becomes

\[
P_{\text{TE}_{10}} = \frac{ab}{2} \sqrt{\frac{\mu f^2}{\epsilon f_0^2}} \sqrt{1 - f^2} |H_0|^2.
\]

It is very instructive to evaluate this power for a specific case. Let the frequency be \( f = 10 \) GHz, and let \( a = 2 \) cm and \( b = 1 \) cm. The cutoff frequency for this waveguide for the \( \text{TE}_{10} \) mode is \( f_c = c/2a = 3 \times 10^8 / 0.04 = 7.5 \) GHz. Let us calculate the maximal possible power that can be transmitted through this waveguide if it is filled with air of dielectric strength \( E_{\text{max}} = 30 \text{kV/cm} \). The maximal electric field is at the coordinate \( x = a/2 \). At that point, the electric field amplitude has to be less than \( E_{\text{max}} \). This enables us to calculate the maximal \( H_0 \) from Eq. (23.27):

\[
H_{0\text{max}} = \frac{\pi E_{\text{max}}/\sqrt{2}}{\omega \mu a}.
\]
Substituting this value of $H_0$ into Eq. (23.32), we obtain that the maximal power that can be transmitted is about 800 kW. This is much more than the power that could be transmitted through a coaxial cable or printed line (why?), so waveguides are the guiding medium of choice for high-power applications, such as some radars.

**Questions and problems:** Q23.8 to Q23.22, P23.5 to P23.10

### 23.5 The Microstrip Line (Hybrid Modes)

We have discussed in detail only one of the TE and briefly one of the TM modes in a rectangular waveguide. There are an infinite number of other TE and TM modes in such guiding structures. However, other structures can support wave types that are a combination of TEM, TE, and TM modes. These wave types are called **hybrid modes**.

As an illustration, we consider a commonly used waveguide, called a **microstrip line**, sketched in Fig. 23.8. A microstrip line is made on a dielectric slab, called the **substrate**. One side of the substrate is coated with metal and acts as the ground electrode, similar to the outer conductor of a coax. A metal strip on the other side of the substrate enables a wave to propagate mostly in the dielectric. The role of the strip is similar to that of the center conductor of a coax. Unlike the coax, however, this structure has an inhomogeneous dielectric. The fields are not contained completely in the dielectric but are partly in air, as sketched in Fig. 23.8. This electric field is usually referred to as a fringing field.

Because there are two conductors in this guide, according to Example 23.2 we may suspect that a microstrip can support a TEM wave. Let us check if this is possible.

The boundary condition for the (fringing) tangential electric field tells us that $E_{x,\text{dies}} = E_{x,\text{air}}$. We replace $E_x$ with spatial derivatives of $\mathbf{H}$ from Maxwell's second equation in differential form, $\mathbf{E} = -\mu \partial \mathbf{H}/\partial t$. Taking into account the boundary condition for the magnetic flux density vector (normal components equal on the two sides of

![Figure 23.8 Sketch of a microstrip line. The electric field lines are sketched in solid line, and the magnetic field lines in dashed line.](image)
the boundary), the boundary condition for the electric field can be written as (see problem P23.11)

$$\varepsilon_r \frac{\partial H_{z,\text{air}}}{\partial y} - \frac{\partial H_{z,\text{dielectric}}}{\partial y} = (\varepsilon_r - 1) \frac{\partial H_y}{\partial z}. \quad (23.33)$$

Let us see what this boundary condition tells us. The right-hand side is not zero, because $\varepsilon_r > 1$, and $H_y$ is not zero. That means that the left-hand side is not zero, which means that $H_z$ is not zero. It can be shown in a similar manner that $E_z$ cannot be zero. So if Maxwell’s equations are satisfied for this structure, the wave type that propagates has to have nonzero $H_z$ and $E_z$ components, which means it contains TE and TM modes, and is a hybrid mode.

The components of the electric and magnetic field vectors along the direction of propagation are small compared to the other components, and this structure supports a wave type referred to as a “quasi-TEM” mode, similar to a TEM mode. This means that we define a characteristic impedance and a propagation constant, and then use TEM mode, or transmission-line equations. These line parameters are expressed in terms of an effective dielectric constant, which depends on the relative permittivity and thickness of the substrate (see problems P23.12 and P23.13).

Microstrip lines are used extensively at microwave frequencies because of their small size and ease of manufacturing (using printed-circuit board technology). Their loss is higher than that in waveguides, so they are not used for high power levels.

Questions and problems: Q23.23 and Q23.24, P23.11 to P23.13

23.6 Electromagnetic Resonators

Classical resonant circuits with lumped elements cannot be used above about 100 MHz. On one hand, losses due to skin effect and dielectric losses become very pronounced. On the other hand, the circuit needs to be sufficiently small not to radiate energy. Therefore at high frequencies, instead of resonant circuits, closed (usually air-filled) metallic structures are used, inside which the electromagnetic field is excited to oscillate. Between about 500 MHz and 3 GHz, resonators are usually in the form of shorted segments of shielded transmission lines (e.g., coaxial line, shielded two-wire line). From about 3 GHz to a few tens of GHz, metallic boxes of various shapes are often used instead (most often in the form of a parallelepiped or circular cylinder). Such boxes are known as cavity resonators. We have seen in Example 22.1 that at still higher frequencies we use so-called Fabry-Perot resonators, consisting of two parallel, highly polished metal plates.

The basic parameters of an electromagnetic resonator are its resonant frequency, $f_r$, the type of wave inside it, and its quality factor, $Q$.

The resonant frequency and the type of wave depend on the resonator shape, size, and excitation, while the quality factor can be defined in general terms. It is defined as the ratio of the electromagnetic energy contained in the resonator, $W_{\text{em}}$, and the total energy lost in one cycle $W_{\text{lost/cycle}}$ in the resonator containing this energy, multiplied by $2\pi$. Since the cycle duration at resonance is $T_r = 1/f_r$, where $f_r$ is the
resonant frequency of the resonator, \( W_{\text{lost/cycle}} = P_{\text{losses}} \cdot T_r = P_{\text{losses}} / f_r \). So the \( Q \) factor can be written in the following two equivalent forms:

\[
Q = 2\pi \frac{W_{\text{em}}}{W_{\text{lost/cycle}}} = \frac{\omega_r}{P_{\text{losses}}} \quad \text{(dimensionless)}
\]

(General definition of \( Q \) factor of electromagnetic resonators)

**Example 23.10—\( Q \) factor of an LC circuit.** The general definition of the \( Q \) factor is valid for resonant circuits also. Consider a parallel connection of a capacitor of capacitance \( C \) and a coil of inductance \( L \). Let the (small) series resistance of the circuit be \( R \). Provided that losses are small, we know that the angular resonant frequency of the circuit is \( \omega_r = 1/\sqrt{LC} \). We also know that energy oscillates between that in the capacitor and that in the coil. When the energy is completely in the coil, the current in the circuit is maximal, for example, \( I_m \). The magnetic energy contained in the coil at that instant is the energy of the circuit, and is simply

\[
W_{\text{em}} = \frac{1}{2} LI_m^2.
\]

*Time-average* Joule’s losses in the circuit corresponding to this current *amplitude* are

\[
P_{\text{losses}} = \frac{1}{2} R I_m^2.
\]

Thus the circuit \( Q \) factor is \( Q = \omega_r L/R \), as defined in circuit theory. It is almost impossible to obtain a resonant circuit with a \( Q \) factor greater than about 100. (Why do you think this is so? See question Q23.25.)

### 23.6.1 TRANSMISSION-LINE SEGMENTS AS ELECTROMAGNETIC RESONATORS

Consider a two-conductor lossless transmission line segment of length \( \xi \) shorted at its end. We assume the \( \xi \) axis to be directed from the shorted end toward the generator, as in Chapter 18. The input impedance of the segment was derived in Example 18.6:

\[
Z(\xi) = jZ_0 \tan(\beta \xi) = jZ_0 \tan \left( \frac{2\pi}{\lambda} - \xi \right), \quad \lambda = \frac{c}{f}, \quad c = \frac{1}{\sqrt{LC}}.
\]

We see that \( Z(\xi) = 0 \) for \( \xi = n\lambda/2, \ n = 1, 2, \ldots \). This means that a shorted transmission line connected to a generator can be short-circuited at any such point (cross section) without affecting the voltage and current along the line. We can even cut off such a section (shorted at both ends) of the *excited* line, and the current and voltage along it will not be affected. Thus, we obtained an electromagnetic resonator of length \( x = n\lambda/2 \).

Assume that the rms value of the voltage of the forward wave is \( V_+ \). The rms value of the forward current wave is \( I_+ = V_+/Z_0 \). We know from Example 18.3 that the voltage reflection coefficient in this case is \( -1 \). Therefore, the voltage and current distribution along the line segment, given in Eqs. (18.22), in this case become

\[
V(\xi) = j2V_+ \sin \beta \xi, \quad I(\xi) = 2I_+ \cos \beta \xi,
\]
which represent standing voltage and current waves. (Note that in these equations we needed to replace $z$ by $-\xi$.) We see that the phase difference between the two is $\pi/2$, which means that the voltage is zero everywhere when the current is maximum, and vice versa. Thus we have the same situation as in resonant circuits: when, for example, the current in the resonator is maximal, the entire energy is in the magnetic field. Such resonators can be made with any transmission line, e.g., twin lead, coaxial cable, or microstrip line. The next example discusses a coaxial cable resonator.

**Example 23.11—Q factor of a coaxial resonator.** The energy in a segment $\lambda/2$ of a coaxial transmission line is

$$W_{\text{em}} = \frac{1}{2} L' |I(\xi)|^2 \sqrt{2} \, d\xi = \frac{1}{2} L' \frac{8 I_0^2 \lambda}{4} = L' I_0^2 \lambda. \quad (23.36)$$

If the conductors have a small resistance $R'$ per unit length, the time-average power losses in the resonator are

$$P_{\text{losses}} = \int_0^{\lambda/2} R'|I(\xi)|^2 \, d\xi = R' \int_0^{\lambda/2} 4 I_0^2 \cos^2(\beta \xi) \, d\xi = R' I_0^2 \lambda. \quad (23.37)$$

According to the definition of the $Q$ factor, we obtain that for such a resonator

$$Q = \frac{\omega L'}{R'}, \quad (23.38)$$

which is of the same form as for a resonant circuit.

This simple theory is valid for all transmission lines. At higher frequencies, however, in open structures like two-wire lines, there may be significant losses due to radiation. Therefore resonators of this type are most often made of a coaxial-line segment, as in Fig. 23.9. The resonator may be excited (and the energy from it extracted) by either a small probe $a$ or a small loop $b$. The probe and the loop are positioned at the voltage maximum (i.e., the electric-field maximum), namely the current maximum (i.e., the magnetic-field maximum).

Figure 23.9 Sketch of a resonant ($\lambda/2$) section of a coaxial line shorted at both ends. The resonator may be excited by a small probe $a$, or a small loop $b$. Note the positions of the two excitation elements.
As an example, consider a coaxial resonator with the inner conductor of radius $a = 0.5$ cm, the inner radius of the outer conductor $b = 1.5$ cm, made of copper $(\sigma = 56 \cdot 10^6$ S/m, $\mu = \mu_0)$, at a frequency $f = 1$ GHz. Using the data from Table 18.1 and Eq. (23.38), we obtain that $Q = 4614$. This is a very large value compared with those for resonant circuits (as mentioned, at the most about 100).

### 23.6.2 RESONANT CAVITIES

Resonant cavities may be of diverse shapes, but we will analyze the simplest, in the form of a parallelepiped (rectangular box). It can be obtained by introducing appropriate short circuits (transverse metallic walls) into a rectangular waveguide.

Consider a shorted rectangular waveguide, as in Fig. 23.10. Let a distant generator at left (not shown) excite in the waveguide the dominant, TE$_{10}$ mode. The wave is reflected at the short circuit, giving rise to a backward wave. The backward wave is of the same form as the forward wave, except that the phase coefficient is now $-\beta$. At the short circuit the $E_y$ component of the reflected wave must have the opposite phase with respect to the incident wave. From Eqs. (23.25) to (23.28) we thus obtain the following expressions for the resulting field in the shorted waveguide:

\[
E_y^{\text{res}}(x, y, z) = j\omega \mu \frac{a}{\pi} H_0 \sin \left( \frac{\pi x}{a} \right) \left( e^{j\beta z} - e^{-j\beta z} \right)
= -2\omega \mu \frac{a}{\pi} H_0 \sin \left( \frac{\pi x}{a} \right) \sin \beta z,
\]  

(23.39)

\[
H_x^{\text{res}}(x, y, z) = 2j\beta \frac{a}{\pi} H_0 \sin \left( \frac{\pi x}{a} \right) \cos \beta z,
\]  

(23.40)

\[
H_z^{\text{res}}(x, y, z) = -2jH_0 \cos \left( \frac{\pi x}{a} \right) \sin \beta z.
\]  

(23.41)

We see that the factor $e^{\pm j\beta z}$ is not present, so we have a standing wave in the waveguide. The other components are zero.

![Figure 23.10 Shorted rectangular waveguide. Indicated in dashed lines is another "short circuit" of the waveguide, resulting in a resonant cavity in the form of a rectangular box.](image-url)
According to Eqs. (23.39) and (23.41), the total \( y \) component of the electric field, and the total \( z \) component of the magnetic field, are zero not only at \( z = 0 \) (at the shorted end), but also in planes \( z = -\pi p/\beta = -p\lambda_z/2, \ p = 1, 2, \ldots \). Consequently, if we place a thin metal foil in any of these planes and thus short the waveguide at one more place, the field will not change, since the boundary conditions are automatically satisfied in these planes. So we obtain a standing wave in a rectangular box of sides \( a, b, \) and \( p\lambda_z/2 \). This type of standing wave is known as the TE\(_{10p}\) mode in the cavity.

**The TE\(_{101}\) Mode**

Let us consider the simplest mode, the TE\(_{101}\) mode. Let \( \lambda_z/2 = d \) in Fig. 23.10. We then have \( \beta = 2\pi /\lambda_z = \pi /d \), so that Eqs. (23.39) to (23.41) become

\[
E_y^{\text{res}}(x, y, z) = -2\omega \mu \frac{a}{\pi} H_0 \sin \left( \frac{\pi}{a} x \right) \sin \left( \frac{\pi}{d} z \right), \\
H_x^{\text{res}}(x, y, z) = 2j\frac{a}{d} H_0 \sin \left( \frac{\pi}{a} x \right) \cos \left( \frac{\pi}{d} z \right), \\
H_z^{\text{res}}(x, y, z) = -2j H_0 \cos \left( \frac{\pi}{a} x \right) \sin \left( \frac{\pi}{d} z \right). \tag{23.42}
\]

From these equations we can deduce how electromagnetic oscillations in the cavity are maintained. There are induced electric charges on the upper and lower cavity walls because the normal component \( E_y^{\text{res}} \) of the electric field vector exists there. Surface currents in the \( y \) direction appear only on the side walls, where \( H_x^{\text{res}} \) and \( H_z^{\text{res}} \) are nonzero. So the oscillations of the electromagnetic field inside the cavity are accompanied by charges and currents on its walls.

According to Eqs. (23.42) to (23.44), the electric and magnetic fields are shifted in phase by \( \pi/2 \) (the factor \( j \)). Therefore, at some points in time there are no currents flowing in the walls, and at others there are no charges. These instants are separated in time by \( T/4 \), where \( T = 1/f \) is the period of the oscillations. Figure 23.11 shows the distribution of charges and currents during one period, starting at the instant when the upper face carries the maximum positive charge.

To determine the quality factor of the cavity, we need to calculate the energy contained in the cavity and the time-average power of losses in the cavity walls corresponding to this energy. At this introductory level, we will just give a numerical example: for a cubic cavity \( (a = b = d) \) filled with air, designed to operate at 3 GHz in the TE\(_{101}\) mode, we find that the side length of the cube is \( a = c/(f\sqrt{2}) = 7.07 \) cm. (Of course, in that case the TE\(_{101}\) mode will be the same as the TE\(_{110}\) or TE\(_{011}\) modes.) Let the cavity be made of copper \( (\sigma = 56 \cdot 10^6 \text{ S/m}, \ \mu = \mu_0) \). Using the losses in the surface resistance of the cavity walls, one can calculate that \( Q \simeq 19,200 \). To obtain this value, the surface resistance is calculated assuming a perfectly polished wall, i.e., with unevenness much less than the skin depth. So as frequency increases, it becomes more and more difficult to avoid increases in loss.

*Questions and problems:*  Q23.25 to Q23.32, and P23.14 to P23.16
23.7 Chapter Summary

1. Electromagnetic waves can be guided along a desired route not only by transmission lines but also by hollow pipes, dielectric-coated surfaces, or dielectric rods.

2. It is typical of all these hollow waveguides that (1) they cannot support the TEM wave type, but support the TE and TM wave types, and (2) they cannot transmit energy below a certain frequency, known as the cutoff frequency.

3. Waves of frequencies above the cutoff frequency propagate without attenuation (except due to losses in the materials).

4. Waves of frequencies lower than the cutoff frequency, known as evanescent modes, are exponentially attenuated, and do not propagate at all.

5. Among hollow metallic waveguides, the most important are those of rectangular cross section. There are an infinite number of both \(TE_{mn}\) and \(TM_{mn}\) modes that can propagate along such waveguides.

6. The higher-order modes (with larger \(m\) and \(n\) values) for the same waveguide must be of increasingly higher frequencies. So there is a frequency range in which only one mode, the \(TE_{10}\) mode, can propagate. This mode is therefore termed the dominant mode.

7. Commonly used printed microstrip lines support a hybrid mode, consisting of both TE and TM wave types. However, because the longitudinal components of the electric and magnetic field vectors are small, we can approximately treat the wave as a "quasi-TEM" wave, similar to that in a transmission line.

8. Electromagnetic resonators support oscillating electromagnetic fields. At high frequencies, two types of such resonators are mostly used, the coaxial-line (or
other similar line) resonator and the cavity-type resonator. The latter is obtained as a special case of a waveguide, short-circuited at two ends.

9. The quality factor of waveguide resonators may be by two orders of magnitude greater than that of lumped-element resonant circuits.

**QUESTIONS**

Q23.1. Write the instantaneous value of $E_{\text{tot}}(x, y, z) = E(x, y)e^{-\gamma z}$, where $\gamma = \alpha + j\beta$.

Q23.2. Complete the derivation of Eq. (23.7).

Q23.3. Define in your own words the TEM, TE and TM waves. What does “mode” mean?

Q23.4. Can the complex propagation coefficient $\gamma$ in Eq. (23.5) be real? Can it have a real part?

Q23.5. What are eigenvalues (characteristic values) of a parameter in a boundary-value problem? What do they depend on?

Q23.6. The wave impedance of a TEM wave is always real. Are the wave impedances of TE and TM waves also always real? Explain.

Q23.7. Under which conditions is the relation (23.13) valid?

Q23.8. What is the physical meaning of the coefficients $m$ and $n$ in the field components inside a rectangular waveguide in Eqs. (23.16) to (23.20)?

Q23.9. What is the phase and group velocity in a rectangular waveguide in these three cases? (1) $f < f_c$, (2) $f = f_c$, and (3) $f > f_c$.

Q23.10. What is the attenuation constant in a rectangular waveguide in these three cases? (1) $f < f_c$, (2) $f = f_c$, and (3) $f > f_c$.

Q23.11. What are the parameters that determine the cutoff frequency in a waveguide?

Q23.12. A signal consisting of frequencies in the vicinity of a frequency $f_1$, and a signal consisting of frequencies in the vicinity of a frequency $f_2$, propagate unattenuated along a rectangular waveguide in the TE$_{10}$ mode. If $f_1 < f_2$, which is faster?

Q23.13. What will eventually happen with the signals from the preceding question if the waveguide is long?

Q23.14. A signal consisting of frequencies in the vicinity of a frequency $f_1$ propagates along a rectangular waveguide as a TE$_{10}$ mode. What happens if the bandwidth of the signal is relatively large?

Q23.15. What are propagating modes and evanescent modes in a waveguide?

Q23.16. You would like to have openings for airing a shielded room (a Faraday’s cage) without enabling electromagnetic energy to enter or leave the cavity. You are aware that a field of a certain microwave frequency is particularly pronounced around the room, but you do not know its polarization. Can you make the openings in the form of waveguide sections? What profile of the waveguide would you use?

Q23.17. You are using a square waveguide that is bent and twisted along its way. The waveguide is excited with the TE$_{10}$ mode (the $E$ field parallel to the $y$ axis). Can you be certain about the polarization of the wave at the receiving point? Explain.

Q23.18. What is the physical meaning of the dominant mode in a waveguide?
Q23.19. A rectangular waveguide along which waves of many frequencies and modes propagate is terminated in a large metal box. Can you extract from the box a signal of a specific frequency and a desired mode by connecting a section of the same waveguide at another point of the box?

Q23.20. How would you construct a high-pass filter (i.e., a filter transmitting only frequencies above a certain frequency), using sections of rectangular waveguides?

Q23.21. Propose a method for exciting the TE_{11} mode in a rectangular waveguide.

Q23.22. You would like for a rectangular waveguide with a TE_{10} wave to radiate (leak) from a series of narrow slots you made in its walls. For this, you need slots that would force the internal waveguide currents to appear on its outer surface. How do the slots need to be oriented to accomplish this?

Q23.23. Sketch the electric and magnetic field lines for two microstrip lines, one with a substrate twice the thickness of the other, but with the same permittivity. In which case is the quasi-TEM approximation more accurate? Explain.

Q23.24. Sketch the electric and magnetic field lines for two microstrip lines on substrates of equal thicknesses, but where one has a permittivity two times higher than the other. In which case is the quasi-TEM approximation more accurate? Explain.

Q23.25. In the resonant circuit of Example 23.10, explain why it is hard to achieve a large Q factor. Why do losses go up as the frequency increases?

Q23.26. You would like to have a coaxial-line resonator with as large a Q factor as possible for a given outer resonator size. What would you do?

Q23.27. Propose two methods for the excitation and energy extraction from a coaxial resonator.

Q23.28. Find the energy contained in coaxial resonators of lengths \( \lambda, \frac{3}{2} \lambda, \) and \( 2\lambda \), using Eq. (23.36). What is the Q factor of these resonators?

Q23.29. Sketch the current and voltage along an open-ended microstrip line resonator that is half of a guided wavelength long. What is the impedance at the center of the resonator, and what at the two ends?

Q23.30. What loss mechanisms can you think of in an open-ended microstrip line resonator?

Q23.31. A rectangular waveguide with a TE_{10} mode is terminated in a large rectangular cavity (e.g., of a microwave oven). Describe qualitatively what happens.

Q23.32. Propose two methods for the excitation and energy extraction from a cavity resonator with a TE_{101} wave type in it.

PROBLEMS

P23.1. Prove that for any TEM wave the electric and magnetic field vectors are normal to each other at all points.

P23.2. Prove that at any cross section of a two-conductor transmission line with a forward traveling wave, the ratio of the voltage between the conductors and the current in them equals \( Z_{\text{TEM}} \) in Eq. (23.7). Show that for two-conductor transmission lines, \( C/L = \epsilon/\mu \).

P23.3. Prove that since at any cross section of a multiconductor transmission line \( \sum Q(z) = 0 \), it follows that \( \sum I(z) = 0 \), where the sum refers to all the conductors of the line.
P23.4. Prove that Eqs. (23.1) to (23.4) imply that the electric and magnetic field vectors of a TE wave are normal to each other at all points.

P23.5. Write the instantaneous values of all the components of the TE_{10} wave in a rectangular waveguide. From these equations, sketch the distribution of the E-field and the H-field in the waveguide at $t = 0$.

P23.6. Determine the cutoff frequencies of an air-filled waveguide with $a = 2.5$ cm and $b = 1.25$ cm, for the following wave types: (1) TE_{01}, (2) TE_{10}, (3) TE_{11}, (4) TE_{21}, (5) TE_{12}, and (6) TE_{22}.

P23.7. Plot the mode impedances between 8 and 12 GHz for an air-filled rectangular waveguide with $a = 2.5$ cm and $b = 1.25$ cm, for the following wave types: (1) TE_{01}, (2) TE_{10}, (3) TE_{11}, (4) TE_{21}, (5) TE_{12}, and (6) TE_{22}.

P23.8. Plot the wavelength $\lambda_z$ along a rectangular waveguide with $a = 2$ cm, $b = 1$ cm, and air as the dielectric, if the wave is of the TE_{10} type, for frequencies between 8 and 10 GHz. Is the wavelength shorter or longer than in an air-filled coaxial line?

P23.9. Plot the phase and group velocities in problem P23.8.

P23.10. In a rectangular waveguide from problem P23.8, two signals are launched at the same instant. The frequency range of the first is in the vicinity of $f_1 = 10$ GHz, and of the second in the vicinity of $f_2 = 12$ GHz. Both signals propagate as TE_{10} waves. Find the time intervals the two signals need to cover a distance $L = 10$ m, and the difference between the two intervals. Which signal is faster?

P23.11. Derive Eq. (23.33) starting from the tangential electric field boundary condition.

P23.12. The effective dielectric constant of a microstrip line depends on its dimensions approximately as

$$\varepsilon_e = \frac{\varepsilon_r + 1}{2} + \frac{\varepsilon_r - 1}{2} \frac{1}{\sqrt{1 + 12h/w}},$$

where the parameters are explained in Fig. P23.12. Plot the effective dielectric constant for $h/w$ ratios between 0.1 and 10 (this is the approximate range for practical use), and for substrates that have relative permittivities of 2.2 (Teflon-based Duroid), 4.6 (FR4 laminate), 9 (aluminum nitride), 12 (high-resistivity silicon), and 13 (gallium arsenide).

Figure P23.12 A microstrip line

P23.13. The approximate formulas for microstrip line impedance and propagation constant based on the quasi-TEM approximation are given by
\[
\beta = \omega \sqrt{\varepsilon_0 \mu_0} \sqrt{\varepsilon_r},
\]

\[
Z_0 = \begin{cases} 
\frac{60}{\sqrt{\varepsilon_r}} \ln \left( \frac{8h}{w} + \frac{w}{4h} \right), & \frac{w}{h} \leq 1 \\
\frac{120\pi}{\sqrt{\varepsilon_r} \left( \frac{(w/h) + 1.393 + 0.667 \ln[(w/h) + 1.444]}{w/h} > 1 \right)}
\end{cases}
\]

Plot the characteristic impedance as a function of the ratio \( w/h \) (between 0.1 and 10), and for the relative permittivities from problem P23.12. What can you conclude about the impedance as the line gets narrower?

P23.14. Plot the current, voltage, and impedance along a half-wavelength coaxial resonator short-circuited at both ends. If you want to feed the resonator with another piece of the same kind of cable, at which place along the resonator would you do it and why?

P23.15. Plot the current, voltage, and impedance along a half-wavelength coaxial line resonator open-circuited at both ends. You want to feed the resonator with a 50-Ω coaxial line. Propose (sketch) a way to do it, and explain.

P23.16. Determine the maximum possible energy stored in a cubical resonant air-filled cavity with \( a = b = d = 10 \text{ cm} \), at a resonant frequency corresponding to the \( \text{TE}_{101} \) wave. The electric strength of air is 30 kV/cm.