2.1 Scattering Parameters (S-Parameters)

In a coax, voltages and currents do make sense, but, for example, in a waveguide, they do not. We mentioned also that it is hard to measure voltages and currents in a transmission line, since any probe presents some load impedance, which changes what we are measuring. The standard quantities used at microwave frequencies to characterize microwave circuits are wave variables and scattering parameters. Usually, in a microwave circuit, we talk about ports, which are not simple wires, but access points of transmission lines or waveguides connected to a circuit. These transmission lines support waves traveling into and out of the circuit. This is shown in Fig. 2.1. A microwave circuit, in general, can have many ports. You can think of this in the following way: we send a wave into an unknown N-port circuit, and by measuring the reflected wave at the same port (like an echo) and the transmitted wave at some other port, we can find out what the microwave circuit is. The problem is the following. Let us say you look at an N-port, you send a wave into port 1, and you look at what gets reflected at port 1 and transmitted, say, at port 3. The result of your measurement will depend on what loads were connected to all of the other ports during the measurement. The convention is to terminate all other ports with the characteristic impedances of the transmission lines connected to the ports, so that there is no reflection from these ports. In other words, you can think of an S-parameter as a generalized reflection or transmission coefficient when all other ports of a multi-port circuit are matched.

Let us look at the generalized N-port microwave circuit in Fig. 2.1. The ports are indexed by the subscript $i$ that goes from 1 to $N$. The normalized voltage waves $a_i$ and $b_i$ are defined as

$$a_i = \frac{V_i^+}{\sqrt{Z_{0i}}} \quad b_i = \frac{V_i^-}{\sqrt{Z_{0i}}}$$

(2.1)

where $V_i^\pm$ are RMS voltages and $Z_{0i}$ is a real normalizing impedance, usually chosen to be the characteristic impedance of the transmission line connected to port $i$ (it is thus assumed hereafter that the externally connected transmission lines have real characteristic impedances; some of the results derived under this assumption are not true if $Z_{0i}$ is complex). The $a_i$’s and $b_i$’s are complex numbers and are often called wave amplitudes. The waves going into the circuit are called incident, and the ones coming out are called scattered.

The magnitudes of $a_i$ and $b_i$ are related to power in the following way. The total currents and voltages expressed in terms of $a_i$ and $b_i$ are

$$V_i = (a_i + b_i)\sqrt{Z_{0i}} \quad , \quad I_i = \frac{a_i - b_i}{\sqrt{Z_{0i}}}.$$  

(2.2)
while conversely,

\[ a_i = \frac{1}{2} \left( \frac{V_i}{\sqrt{Z_{0i}}} + I_i \sqrt{Z_{0i}} \right), \quad b_i = \frac{1}{2} \left( \frac{V_i}{\sqrt{Z_{0i}}} - I_i \sqrt{Z_{0i}} \right) \]  

(2.3)

Since we are using RMS quantities, the power going into port \( i \) is equal to

\[ P_i = Re \{ V_i I_i^* \} = Re \{ (a_i + b_i)(a_i - b_i)^* \} = |a_i|^2 - |b_i|^2, \]  

(2.4)

where the asterisk denotes the complex conjugate of a complex number. This formula means that we can interpret the total power going into port \( i \) as the incident power \( |a_i|^2 \) minus the scattered power \( |b_i|^2 \). This formula can be extended to calculate the power flowing into the entire circuit:

\[ P_{IN} = \sum_i P_i = a^\dagger a - b^\dagger b, \]  

(2.5)

where \( a^\dagger \) is the Hermitian conjugate, that is the complex conjugate of \( a \) transposed. Here \( a \) is a column vector of order \( N \) consisting of all the \( a_i \)'s. Usually this is defined as an input vector, and the vector \( b \) is defined as the output vector of a microwave network, and they are related by

\[ b = Sa, \]  

(2.6)

where \( S \) is called the scattering matrix.

In principle, we can measure the coefficients of the scattering matrix by terminating all the ports with their normalizing impedance, and driving port \( j \) with an incident wave \( a_j \). All the other \( a_k \) waves
will be zero, since the other terminations are matched and have no reflection. The scattering coefficients \( S_{ij} \) are then

\[
S_{ij} = \frac{b_j}{a_j} \bigg|_{a_k=0} \text{ for } k \neq j.
\]  

(2.7)

As an example, for a typical two-port network as shown in Fig. 2.2, the scattering matrix is a \( 2 \times 2 \) matrix, the scattering coefficient \( S_{11} \) is the reflection coefficient at port 1 with port 2 terminated in a matched load, and the scattering coefficient \( S_{21} \) is the transmission coefficient from port 1 to port 2. Mathematically, we have

\[
\begin{align*}
S_{11} & = \frac{b_1}{a_1} \quad \text{when } a_2 = 0, \\
S_{21} & = \frac{b_2}{a_2} \quad \text{when } a_1 = 0.
\end{align*}
\]

(2.8)

Figure 2.2: A two-port microwave network. The waves \( a_1 \) and \( a_2 \) are the input waves, the waves \( b_1 \) and \( b_2 \) are the output waves, and \( S \) is the scattering matrix for this network. One example of a two port network is just a section of transmission line.

\[
b_1 = S_{11}a_1 + S_{12}a_2, \quad b_2 = S_{21}a_1 + S_{22}a_2
\]

(2.8)

where, following (2.7), we have

\[
S_{11} = \frac{b_1}{a_1} \quad \text{when } a_2 = 0.
\]

(2.9)

and so forth.

When modeling \( S \)-parameters of a network in a circuit analysis program such as SPICE which does not natively include the capability of handling incident and reflected waves, the following trick is often useful. Consider the circuit of Figure 2.3(a), where a load is connected at the end of a transmission line of characteristic impedance \( Z_0 \) on which an incident voltage wave \( v_+ \) is present. We can use the Thévenin equivalence theorem to replace the transmission line by an equivalent generator and equivalent Thévenin impedance. The generator voltage is found by open-circuiting the ends, and is equal to \( v_{Th} = 2v_+(t) \) because the reflection coefficient of an open circuit is \( -1 \). The short-circuit current is \( v_+(t)/Z_0 \) because the current reflection coefficient of a short is \( -1 \) (the negative of the voltage reflection coefficient). Therefore, the Thévenin equivalent circuit of a lossless transmission line terminated in some load is shown in Fig. 2.3(b) and is given by

\[
Z_{Th} = \frac{2v_+}{2v_+/Z_0} = Z_0, \quad \text{and} \quad V_{Th} = 2v_+.
\]

(2.10)

This works in the frequency domain, and also in the time domain if the characteristic impedance \( Z_0 \) is equal to a frequency-independent resistance, as will be the case for a lossless line.

### 2.2 Reciprocal and Lossless Networks

In general, a scattering matrix has many parameters that need to be determined for a specific network. For example, a 4-port network has a \( 4 \times 4 \) scattering matrix, and in this case the network is determined
by 32 real numbers (each scattering parameter is complex). Fortunately, in many cases it is possible to reduce the number of unknown coefficients knowing some of the properties of the network. One important property is reciprocity. A network is reciprocal if the power transfer and the phase do not change when the input and output are interchanged. This means that for reciprocal networks, the scattering matrices are symmetrical. In order for a network to be reciprocal, it has to be linear, time invariant, made of reciprocal materials, and there cannot be any dependent voltage or current sources in the network. For example, a transistor amplifier is not reciprocal because of the dependent current source, and you know from your circuits classes that an amplifier usually does not work well backwards. A nonreciprocal device used commonly in microwave engineering is an isolator, which contains a nonreciprocal material called a ferrite. In this case there is a static magnetic field that gives a preferred direction to the device. Isolators typically have a low loss in one direction, about 1 dB, and a very high loss in the other direction, usually about 20 dB or more. Isolators are often used to protect a transmitter, just like the one you will be using at the output of the sweepers in at least one of your lab experiments. For example, if you have a radar that is producing a megawatt (MW), in case of an open circuited output, you do not want the power to reflect back into the transmitter.

*Reciprocal circuits have a symmetrical scattering matrix*, which means that $S_{ij} = S_{ji}$. For example, the scattering matrix of a reciprocal two-port looks like

$$S = \begin{bmatrix} S_{11} & S_{21} \\ S_{21} & S_{22} \end{bmatrix}.$$  \hspace{1cm} (2.11)

Another simplification that can be made in a scattering matrix is when the network is lossless, which means it absorbs no power. This means that the scattered power is equal to the incident power, or mathematically

$$b^H b = a^H a.$$  \hspace{1cm} (2.12)

It turns out that this is equivalent to saying that the scattering matrix is unitary, which is written as

$$S^H S = I,$$  \hspace{1cm} (2.13)

where $I$ is the identity matrix. This means that the dot product of any column of $S$ with the complex conjugate of the corresponding row (or of the same column if the network is reciprocal) gives unity, and the dot product of a column with the complex conjugate of a different column gives a zero. In other
words, the columns of the scattering matrix form an orthonormal set, which cuts down the number of
independent S-parameters by a factor of two. In the case of a lossless reciprocal two-port, (2.13) leads
to the three independent scalar identities:

\[ |S_{11}|^2 + |S_{21}|^2 = 1 \ ; \hspace{1cm} |S_{22}|^2 + |S_{21}|^2 = 1 \ ; \hspace{1cm} S_{11}^* S_{21} + S_{21}^* S_{22} = 0 \ . \]  

(2.14)

which serve to further reduce the number of independent quantities in the scattering matrix.

Example

As a simple example, consider the junction of two transmission lines, one (at port 1) with a characteristic
impedance of \( Z_{01} \), the other (at port 2) with a characteristic impedance of \( Z_{02} \). Nothing else appears in
the circuit. We know that if port 2 is connected to a matched load, the voltage reflection coefficient of
a wave incident at port 1 is

\[ \rho_1 = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \]

and the transmission coefficient is \( \tau_1 = 1 + \rho_1 \). Now we note that

\[ a_1 = \frac{V_1^+}{\sqrt{Z_{01}}}; \hspace{1cm} b_1 = \frac{V_1^-}{\sqrt{Z_{01}}} \]

so that

\[ S_{11} = \frac{b_1}{a_1} = \frac{V_1^-}{V_1^+} = \rho_1 \]

On the other hand,

\[ b_2 = \frac{V_2^-}{\sqrt{Z_{02}}} \]

so that \( S_{21} \) is not quite so simple:

\[ S_{21} = \frac{b_2}{a_1} = \tau_1 \frac{\sqrt{Z_{01}}}{\sqrt{Z_{02}}} = \frac{2 \sqrt{Z_{01} Z_{02}}}{Z_{02} + Z_{01}} \]

If we next consider a wave incident at port 2 with a matched load connected to port 1, we get in a
similar way:

\[ S_{22} = \frac{Z_{01} - Z_{02}}{Z_{02} + Z_{01}}; \hspace{1cm} S_{12} = \frac{2 \sqrt{Z_{01} Z_{02}}}{Z_{02} + Z_{01}} \]

Note that \( S_{12} = S_{21} \), as we should expect for this reciprocal network.

So far, we have only talked about scattering parameters of two port networks. There are many
applications when one might want to use a network with more ports: for example, if there is a need to
split the power in one transmission line or waveguide into several others, or combine the power from
several lines into one.

Let us say we wish to make a network that will have three ports and will be used as a two-way power
splitter (or combiner). We would like this network to be lossless and matched at all ports. Such a circuit
is also reciprocal if it is passive and contains no material anisotropy. From the reciprocity condition, we
know that the scattering matrix has to be symmetrical, and from the matched condition we know that
all three reflection coefficients at the three ports are zero, so the scattering matrix of such a device looks
like:

\[ S = \begin{bmatrix}
0 & S_{12} & S_{13} \\
S_{12} & 0 & S_{23} \\
S_{13} & S_{23} & 0
\end{bmatrix} . \]  

(2.15)
Now we can use the lossless condition, which tells us that the matrix is unitary. This gives us the following equations:

\[ |S_{12}|^2 + |S_{13}|^2 = 1 \quad S_{13}^* S_{23} = 0 \]
\[ |S_{12}|^2 + |S_{23}|^2 = 1 \quad S_{23}^* S_{12} = 0 \]
\[ |S_{13}|^2 + |S_{23}|^2 = 1 \quad S_{12}^* S_{13} = 0. \]  

(2.16)

The last three equations imply that at least two of the three parameters have to be equal to zero. This contradicts the first three equations. The conclusion is that a lossless, reciprocal and matched three-port is not possible.

If one of the three conditions is dropped, a feasible device can be made. For example, if we assume the device is not reciprocal, but is matched and lossless, a device called a circulator results. It has the property that power coming into port 1 will go out of port 2, and none will go out of port 3, power going into port 2 will only go out port 3, and power going into port 3 will only go out port 1. It is now obvious where the name comes from. This device is widely used in microwave engineering, and is physically realized by using ferrite materials, which give it a preferred direction by producing a static magnetic field in only one direction. If a three port is reciprocal and lossless, but not necessarily matched at all ports, we get a power divider, and finally, if we relax the lossless condition, we get a resistive power divider. Such resistive power dividers can be made such that \( S_{23} = S_{32} = 0 \) so that the two output ports are isolated. They can also be designed to have more than 2 outputs. All of these components are often used in microwave circuits.

### 1.8 Microstrip Circuits

In addition to traditional coaxial and waveguide components, many other types of transmission line and waveguide can be used at microwave frequencies. One of the most often used structures today is the microstrip, shown in Fig. 1.16. The wave is guided between the bottom ground plane and the top metal strip. Some of the fields spill over from the dielectric into air. Usually the mode guided in a microstrip circuit is called a quasi-TEM mode, because it almost looks like a TEM mode, but the fields do have a small component in the propagation direction. In reality, the guided mode is a hybrid TE and TM mode, and the analysis is quite complicated. If you imagine that there is no dielectric, just air, the mode could be TEM. The presence of the dielectric complicates things, but people have been able to use quasi-static analysis to obtain formulas for the impedances and propagation constants in microstrip lines. In those formulas, the so called effective dielectric constant is used. This is just some kind of average between the permittivity of air and the dielectric that gives you a rough picture of the portion of field that remains in the dielectric. It is usually found from:

\[
\varepsilon_e = \frac{\varepsilon_r + 1}{2} + \frac{\varepsilon_r - 1}{2} \frac{1}{\sqrt{1 + 12 \frac{h}{w}}} 
\]  

(1.66)

There are many approximate closed-form expressions that have been obtained for microstrip line impedance and propagation constants; they give results usually within one percent of each other, and you can find them in almost any modern microwave textbook. We will just list one such set here for easy reference:

\[
\beta = k_0 \sqrt{\varepsilon_e} \\
Z_0 = \begin{cases} 
\frac{60}{\sqrt{\varepsilon_e}} \ln \left( \frac{8h}{w} + \frac{w}{4h} \right), & \frac{w}{h} \leq 1 \\
\frac{120\pi}{\sqrt{\varepsilon_e} \left[ \frac{w}{h} + 1.393 + 0.667 \ln \left( \frac{w}{h} + 1.444 \right) \right]}, & \frac{w}{h} > 1 
\end{cases} 
\]  

(1.67)
Here are a few practical facts to keep in mind:

- the width-to-height ratio $w/h$ is usually between 0.1 and 10
- the higher the dielectric constant, the thinner the line, keeping the thickness of the dielectric and the impedance of the line constant
- the thinner the dielectric, the thinner the line is, keeping the dielectric constant and impedance of the line constant
- the higher the dielectric constant, the smaller the circuit is (why?)
- the wider the line, the lower the impedance. Impedances in the range from 20 to 125$\Omega$ are typically used (what are the two limits determined by?).
- microstrip has relatively low values for the unloaded Q factor, typically less than a hundred for $\varepsilon_r = 10$.

Microstrip circuits became popular because they are planar (flat), small, easy and fast to make and cheap. However, they cannot handle very high power levels and they are more lossy than coax or waveguide.
The loss limits the useful frequency range above about 40GHz, so in the millimeter-wave range waveguide and quasi-optical circuits are mostly in use.

Microstrip circuits are made on many different types of dielectric substrates. A list of the most commonly used ones follows.

- Ceramic substrates, for example alumina. Permittivities range between 8 and 10. Used up to 20GHz. Relatively good thermal conductivity. Relatively brittle.

- Plastic substrates, for example Duroid (a teflon-based dielectric loaded with alumina particles). Permittivity in the range between 2.2 and 10.5. Used to about 20GHz. Bad thermal conductivity, except for some newer materials, such as the Rogers’ TMM serie. Easy to handle.

- High resistivity semiconductor substrates, GaAs ($\varepsilon_r = 13$) and silicon ($\varepsilon_r = 12$). The semiconductors have to be very pure, the silicon wafers are not standard and are quite lossy. Silicon has a reasonable thermal conductivity, about three times higher than GaAs. Both have low dielectric strength compared to other substrates in use. GaAs can be used for monolithic circuits, since most microwave solid-state devices are made in GaAs (the exception are IMPATT diodes, InP Gunn diodes, and silicon pin diodes). GaAs is very brittle and hard to handle in the thickness needed for MMICs (usually 40μm). Used up to 40GHz.

- Sapphire. Anisotropic, with $\varepsilon_r = 9.4$ and 11.6. Wafers can be very uniform. Good thermal conductor. Optically transparent, easy to handle.

- Fused quartz (low-loss glass). $\varepsilon_r = 3.8$, optically transparent, bad thermal conductor. Very low losses, used up to 300GHz. Easy to handle.

- Aluminum Nitride. A relatively new substrate with $\varepsilon_r = 9$ approximately, brittle, but with excellent thermal conductivity.

- Beryllia (BeO). Highly toxic, should be avoided. Used because of its excellent thermal conductivity. $\varepsilon_r = 6$

- FR4 and other circuit-board low-frequency laminate materials ($\varepsilon_r = 4$ to 5) are a good low-cost alternative for up to around 3GHz, above which their loss becomes too high.

The metal technology can be thick film or thin film. Thick film copper can be rolled or pressed, and the typical fabrication procedure is etching using positive photoresist patterning. Thin film assumes evaporated metal. First a seed layer, usually 200 Angstroms of chromium, is evaporated, and then a layer of, say, gold. The metal needs to be at least 4 skin depths thick, and 4μm at 4GHz is sufficient and is usually electroplated on top of the evaporated metal.

**Some Practical Parameters of Microstrip**

- The metal strip in all the formulas is assumed to be two-dimensional. In reality, it has a finite thickness. This is usually taken into account by using an effective width of the strip, $w_{eff} > w$.

- Dispersion. The permittivity of the substrate is a function of frequency. A qualitative dependance is shown in Figure L1.6(a).

- Losses in microstrip result in attenuation. The different loss mechanisms are: (a) conductor loss (on the order of 0.1dB/wavelength), (b) dielectric loss (on the order of 0.01dB/wavelength), (c) radiation loss (proportional to $hf / \sqrt{\varepsilon_r}$), and (d) surface wave loss. These loss mechanisms limit the Q-factor of microstrip circuits.

- Microstrip transmission lines are frequency limited on the upper end due to the fact that the quasi-TEM mode starts coupling to the lowest order TM mode, or to the lowest order transverse resonance mode, and this causes losses. We can understand the TM mode easily by looking at microstrip in the thick dielectric slab limit (very thin lines). The quasi-TEM mode will couple to the TM lowest order mode when their phase velocities become equal. Since the phase velocities
of the TM mode is a function of the dielectric thickness, from here we can get a dependence of upper frequency limit versus substrate thickness. For thin lines and a thick substrate, the usable substrate thickness as a function of the free-space wavelength is given by

\[
h = \frac{0.345 \lambda_0}{\sqrt{\varepsilon_r - 1}}
\]

At the lowest order transverse resonant mode cutoff, the equivalent circuit of this mode is a resonant transmission line of “length” \( w+2d \), where \( d \) describes the fringing capacitance at the edge of the line, Figure (b). Low impedance lines give rise to this frequency limit.

![Figure](a) The qualitative behaviour of the permittivity as a function of frequency. (b) Equivalent circuit for the lowest order transverse resonant mode in microstrip

- Microstrip discontinuities that cannot be avoided in practical circuits are: open circuits, series coupling gaps, shorts to ground plane, bends, impedance steps and T-junctions. These discontinuities should be accounted for in the design. Please remember them in your lab designs.
- Manufacturing tolerances degrade the VSWR of microstrip circuits, but the VSWR is usually less than 1.4.
- Microstrip circuits are often packaged in metal boxes for shielding purposes, and these packages affect the circuit performance. Often the packages are lined with absorptive material.
2.8 Single-Stub Matching

2.8.1 Smith chart method

A shunt stub is an open or short circuited section of transmission line in shunt with the load and the line that the load is being matched to, shown in Fig. 2.11. The distance $d$ between the stub and load needs to be determined, as well as the length $l$ of the stub, the characteristic impedance of which is $Z_0$. The idea is that the distance $d$ is selected so that the admittance $Y = 1/Z$ looking towards the load at plane $1 - 1'$ is equal to $Y = Y_0 + jB$, and then the stub admittance is chosen to be $-jB$ to tune out the reactive part of the input admittance $Y$, which results in a matched condition. We will solve an example of single-stub matching both analytically and on the Smith chart. Since the stub is in shunt, it is more convenient to use admittances instead of impedances.

Example

Let us match a load impedance of $Z_L = 15 + j10 \Omega$ to a 50 $\Omega$ transmission line using a single shunt stub. We will use the admittance chart, enter the normalized load impedance $z_L = 0.3 + j0.2$, then rotate its locus 180° about the center to obtain the locus of the normalized admittance $y_L = 1/z_L = 2.31 - j1.54$ and the corresponding SWR circle, as shown in Fig. 2.12. The two points of intersection of the SWR circle and the $g = 1$ circle give two possible admittance values $y_1 = 1 - j1.33$ (which we will denote as case 1) and $y_2 = 1 + j1.33$ (which we denote case 2) as shown. This means that the stub susceptance needs to be $b_1 = +1.33$ or $b_2 = -1.33$ and the stub needs to be connected at $d = d_1 = 0.325\lambda - 0.284\lambda = 0.04\lambda$ or $d = d_2 = (0.5 - 0.284)\lambda + 0.171\lambda = 0.387\lambda$ away from the load towards the generator, respectively.

The stub itself can be either open-circuited or short-circuited at its end, as is the more convenient for a particular kind of transmission line. To find the length of an open circuited stub of susceptance $b_1$, we reason in the following manner: we need to end up at $y = 0$ (open circuit), so we move from $y = 0$ along the outer edge of the Smith chart ($g = 0$) to the circle corresponding to $b_1$ by $l_1 = 0.147\lambda$ in case 1. Similarly, the solution for case 2 gives $l_2 = 0.353\lambda$. 

![Figure 2.11: Shunt single-stub matching.](image)
Figure 2.12: Normalized load admittance $y_L$ and stub match calculations on a Smith chart (case 1 calculations shown in red, case 2 in blue).

Do these two matching networks behave in the same way? In order to find that out, let us look at how the two matched circuits described above function as the frequency changes. First we need to say at what frequency we matched the load and what the load is (specifically, how it varies with frequency). Let us assume that the load is matched at 2 GHz and that it is a resistor in series with an inductor. This means that the load impedance is at 2 GHz a resistor of $R = 15 \Omega$ in series with a $L = 0.796 \text{nH}$ inductor. Now we can plot the reflection coefficient at the plane $1 - 1'$ as a function of frequency, which is shown in Fig. 2.13. Case 1 has broader bandwidth than does case 2, and this makes sense since the lengths of the transmission line sections are shorter, so we expect there to be less dependence on wavelength.
**Some Examples of Scattering Parameters**

1. Scattering parameters of two connected transmission lines (2-port)
Two lines of different characteristic impedances are connected, and the connection is a 2-port network. We assume that each line is terminated in its characteristic impedance to find the S-parameters.

\[
\begin{align*}
& a_1 = \frac{V_i^+}{V_{20}}, \quad b_1 = \frac{V_i^-}{V_{20}} \\
& S_{11} = \frac{b_1}{a_1} = \rho_1 = \frac{V_i^-}{V_i^+} = \frac{Z_{20} - Z_{21}}{Z_{20} + Z_{21}} = -S_{21} \\
& S_{21} = \frac{b_2}{a_1} = \frac{V_2^-}{V_{20}} \frac{V_i^+}{V_{20}} = \frac{Z_{20}}{Z_{21}} (1 + \rho_1) \\
& S_{21} = \frac{2\sqrt{Z_{20}Z_{21}}}{Z_{20} + Z_{21}} = S_{12} \\
& S = \begin{bmatrix}
    Z_{20} - Z_{21} & 2\sqrt{Z_{20}Z_{21}} \\
    2\sqrt{Z_{20}Z_{21}} & Z_{20} + Z_{21}
\end{bmatrix}
\end{align*}
\]

2. Scattering parameters of shunt and series elements inserted in a transmission line (2-ports)
The general shunt element has a normalized admittance \( y = l/z \).

\[
\begin{align*}
& y = \frac{Y}{Y_0} = \frac{1}{\delta} \\
& S_{11} = \frac{1 - y}{1 + y} = \frac{1 - (1+y)}{1 + (1+y)} = \frac{-y}{2+y} \\
& S_{21} = \frac{b_2}{a_1} = \frac{a_1 + b_1}{a_1} = 1 + S_{11} = \frac{2}{2+y} \\
& \text{(since element is in shunt, } V_2 = V_1) \\
& S = \frac{1}{2+y} \begin{bmatrix}
    -y & 2 \\
    2 & -y
\end{bmatrix}
\]

Network is reciprocal:
\( S_{ii} = S_{22}, \quad S_{21} = S_{12} \)
3. Scattering parameters of a resistive attenuator (2-port)

A fixed or variable attenuator can be made as shown in the figure below. An attenuator needs to be matched and provide a fixed amount of attenuation in dB.

\[ A = \text{attenuation in dB} \]
\[ A = 10 \log \frac{P_{\text{out}}}{P_{\text{in}}} = 20 \log \frac{V_{\text{out}}}{V_{\text{in}}} \]

\[ S = \begin{bmatrix} 0 & -A/20 \\ -A/20 & 10 \end{bmatrix} \]

\[ S_{21} = 10 \left( \frac{r_1 + 1}{r_1 + r_2 + 1} \right) \]

\[ z_{\text{in}} = \frac{(r_1 + 1)r_2}{r_1 + r_2 + 1} \quad \text{to be matched} \quad (S_{11} = 0) \]

\[ S_{21} = \frac{(r_1 + 1)r_2}{(r_1 + 1)r_2 + 1} \quad \text{gives second relation for given } A(\text{dB}) \]
4. Scattering parameters of a shunt resonant circuit (2-port)
This example is interesting if we consider the change of S-parameters over frequency. In case of the series resonance, it reflects everything at resonance, below resonance, the circuit is inductive, and above resonance it is capacitive. In the case of a parallel circuit, it transmits everything at resonance, and is capacitive below resonance and inductive above resonance.

At resonance, \( f_0 = \frac{1}{2\pi\sqrt{LC}} \)

\( S_{11} = -1 \) (short)
\( S_{21} = 0 = 1 + S_{11} \)
\( S_{22} = S_{11} \), \( S_{21} = S_{12} \)

\[ S\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \]

At resonance,
\( S_{11} = 0 \) (open) \( S_{22} = 0 \)
\( S_{21} = 1 = 1 + S_{11} = S_{12} \)

From #2, you can find values for other \( f \)

This is what you would measure.

5. Scattering parameters of an ideal isolator (2-port)
An isolator is a device that is matched but has a preferred direction of transmission. It is used to protect devices from excessive reflections, e.g., it is used at the output of a transmitter in case the antenna reflection coefficient changes.

\[ \begin{align*}
\begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}
\end{align*} \]

\[ \begin{align*}
\text{Is it lossless? Reciprocal?}
\end{align*} \]
6. **Scattering parameters of an ideal transformer (2-port)**

If the turns ratio is $n$, the impedance seen from the primary is $n^2 Z$ since the voltage is $n$ times larger and the current is $n$ times smaller on the secondary.

\[
\begin{align*}
S_{11} &= \frac{n^2 - 1}{n^2 + 1} \\
S_{12} &= \frac{n^2}{n^2 + 1}
\end{align*}
\]

\[
\begin{align*}
S_{21} &= \frac{b_2}{a_1} |_{a_2=0} = \frac{1}{n} \left( a_1 + b_1 \right) = \frac{1}{n} \left( 1 + S_{11} \right) = \frac{2n}{1 + n^2} \\
S_{22} &= \frac{b_1}{a_2} |_{a_1=0} = \frac{n \left( a_2 + b_2 \right)}{a_2} = n \left( 1 + S_{22} \right) = \frac{2n}{1 + n^2}
\end{align*}
\]

\[
S = \begin{bmatrix}
n^2 - 1 & 2n \\
2n & 1 - n^2
\end{bmatrix}
\]

7. **Scattering parameters of an ideal circulator (3-port)**

An ideal circulator is a device that is connected to an antenna in a transmit-receive system to protect the receiver from transmitter signals. It uses a magnetic component to achieve the non-reciprocity.

\[
S = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

Example use:

- **PA** (transmitter)
- **LNA** (receiver)
- **Antenna**

(protect receiver from transmitter)
8. Scattering parameters of a tee divider (3-port)
The tee divider is shown in the figure, along with the scattering matrix parameters.

\[
\begin{align*}
3m &= \frac{1}{2} \\
(\text{in parallel})
\end{align*}
\]

\[
S_{11} = \frac{\frac{1}{2} - 1}{\frac{1}{2} + 1} = -\frac{1}{3}
\]

\[
S_{21} = 1 + S_{11} = \frac{2}{3}
\]

Check: lossless (Yes),
matched (No),
reciprocal (Yes)

Examples:
- coax Tee
- microstrip Tee

\[
S = \frac{1}{3}
\begin{bmatrix}
-1 & 2 & 2 \\
2 & -1 & 2 \\
2 & 2 & -1
\end{bmatrix}
\]

9. Scattering parameters of a cross (4-port)
The tee divider is shown in the figure, along with the scattering matrix parameters.

\[
\begin{align*}
3m &= \frac{1}{3} \\
S_{11} &= \frac{\frac{1}{3} - 1}{\frac{1}{3} + 1} = -\frac{1}{2}
\end{align*}
\]

\[
S_{21} = 1 + S_{11} = \frac{1}{2}
\]

\[
S = \frac{1}{2}
\begin{bmatrix}
-1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1
\end{bmatrix}
\]
10. Determining if a network is matched, reciprocal or lossless
Matched: diagonal S matrix elements are 0
Reciprocal: matrix S is symmetrical
Lossless: dot product of each column with its complex conjugate is 1, and dot product of any two columns is zero (matrix S is Hermitian)