The preceding problem of modal dispersion in transmission lines is avoided by assuring that only the TEM mode propagates, and that all waveguide modes are below cutoff. This is accomplished either by using line dimensions that are smaller than one-half the signal wavelength or by assuring an upper limit to the operating frequency in a given line. But it is more complicated than this.

In Section 14.1, we saw that increasing the frequency increases the line loss as a result of the skin effect. This is manifested through the increase in the series resistance per unit length, $R$. One can compensate by increasing one or more dimensions in the line cross section, as shown for example in Eqs. (7) and (12), but only to the point at which moding may occur. Typically, the increasing loss with increasing frequency will render the transmission line useless before the onset of moding, but one still cannot increase the line dimensions to reduce losses without considering the possibility of moding. This limitation on dimensions also limits the power-handling capability of the line because the voltage at which dielectric breakdown occurs decreases with decreasing conductor separation. Consequently, the use of transmission lines as frequencies are increased beyond a certain point becomes undesirable, for losses will become excessive, and the limitation on dimensions will limit the power-handling capability. Instead, we look to other guiding structures, among which is the rectangular guide.

The important fundamental difference between the rectangular waveguide (or any hollow pipe guide) and the transmission line is that the rectangular guide will not support a TEM mode. We have already demonstrated this in our study of the TE wave. The fact that the guide is formed from a completely enclosed metal structure means that any electric field distribution in the transverse plane must exhibit variations in the plane; this is because all electric field components that are tangent to the conductors must be zero at the conducting boundaries. Since $E$ varies in the transverse plane, the computation of $H$ through $\nabla \times E = -j\omega\mu H$ must lead to a $z$ component of $H$, and so we cannot have a TEM mode. We cannot find any other orientation of a completely transverse $E$ in the guide that will allow a completely transverse $H$.

Since the rectangular guide will not support a TEM mode, it will not operate until the frequency exceeds the cutoff frequency of the lowest-order guided mode of the structure. Thus, it must be constructed of large enough size to accomplish this for a given frequency; the required transverse dimensions will consequently be larger than those of a transmission line that is designed to support only the TEM mode. The increased size, coupled with the fact that there is more conductor surface area than in a transmission line of equal volume, means that losses will be substantially lower in the rectangular waveguide structure. Additionally, the guides will support more power at a given electric field strength than a transmission line, since the rectangular guide will have a higher cross-sectional area.

Still, hollow pipe guides must operate in a single mode in order to avoid the signal distortion problems arising from multimode transmission. This means that the guides must be of dimension such that they operate above the cutoff frequency of the lowest-order mode, but below the cutoff frequency of the next higher-order mode, as demonstrated in Example 14.4. Increasing the operating frequency again means that the guide transverse dimensions must be decreased to maintain single-mode operation. This can be accomplished to a point at which skin effect losses again
become problematic (remember that the skin depth is decreasing with increasing frequency, in addition to the decrease in metal surface area with diminishing guide size). In addition, the guides become too difficult to fabricate, with machining tolerances becoming more stringent. So again, as frequencies are further increased, we look for another type of structure.

**D14.10.** Specify the minimum width, $a$, and the maximum height, $b$, of an air-filled rectangular guide so that it will operate single mode over the frequency range $15$ GHz $< f < 20$ GHz.

**Ans.** 1 cm; 0.75 cm

### 14.6 PLANAR DIELECTRIC WAVEGUIDES

When skin effect losses become excessive, a good way to remove them is to remove the metal in the structure entirely and use interfaces between dielectrics for the confining surfaces. We thus obtain a **dielectric waveguide**; a basic form, the **symmetric slab waveguide**, is shown in Figure 14.19. The structure is so named because of its vertical symmetry about the $z$ axis. The guide is assumed to have width in $y$ much greater than the slab thickness $d$, so the problem becomes two-dimensional, with fields presumed to vary with $x$ and $z$ while being independent of $y$. The slab guide works in very much the same way as the parallel-plate waveguide, except wave reflections occur at the interfaces between dielectrics, having different refractive indices, $n_1$ for the slab and $n_2$ for the surrounding regions above and below. In the dielectric guide, total reflection is needed, so the incident angle must exceed the critical angle. Consequently, as discussed in Section 13.6, the slab index, $n_1$, must be greater than that of the surrounding materials, $n_2$. Dielectric guides differ from conducting guides in that power is not completely confined to the slab but resides partially above and below.

![Symmetric dielectric slab waveguide structure](image-url)

**Figure 14.19** Symmetric dielectric slab waveguide structure, in which waves propagate along $z$. The guide is assumed to be infinite in the $y$ direction, thus making the problem two-dimensional.
Dielectric guides are used primarily at optical frequencies (on the order of $10^{14}$ Hz). Again, guide transverse dimensions must be kept on the order of a wavelength to achieve operation in a single mode. A number of fabrication methods can be used to accomplish this. For example, a glass plate can be doped with materials that will raise the refractive index. The doping process allows materials to be introduced only within a thin layer adjacent to the surface that is a few micrometers thick.

To understand how the guide operates, consider Figure 14.20, which shows a wave propagating through the slab by multiple reflections, but where partial transmission into the upper and lower regions occurs at each bounce. Wavevectors are shown in the middle and upper regions, along with their components in the $x$ and $z$ directions. As we found in Chapter 13, the $z$ components ($\beta$) of all wavevectors are equal, as must be true if the field boundary conditions at the interfaces are to be satisfied for all positions and times. Partial transmission at the boundaries is, of course, an undesirable situation, since power in the slab will eventually leak away. We thus have a leaky wave propagating in the structure, whereas we need to have a guided mode. Note that in either case, we still have the two possibilities of wave polarization, and the resulting mode designation—either TE or TM.

Total power reflection at the boundaries for TE or TM waves implies, respectively, that $|\Gamma_s|^2$ or $|\Gamma_p|^2$ is unity, where the reflection coefficients are given in Eqs. (71) and (69) in Chapter 13:

$$\Gamma_s = \frac{n_{2s} - n_{1s}}{n_{2s} + n_{1s}}$$  
(89)
and

\[ \Gamma_p = \frac{\eta_{2p} - \eta_{1p}}{\eta_{2p} + \eta_{1p}} \]  

(90)

As discussed in Section 13.6, we require that the effective impedances, \( \eta_{2s} \) or \( \eta_{2p} \), be purely imaginary, zero, or infinite if (89) or (90) is to have unity magnitudes. Knowing that

\[ \eta_{2s} = \frac{\eta_2}{\cos \theta_2} \]  

(91)

and

\[ \eta_{2p} = \eta_2 \cos \theta_2 \]  

(92)

the requirement is that \( \cos \theta_2 \) be zero or imaginary, where, from Eq. (75), Section 13.6,

\[ \cos \theta_2 = \left[ 1 - \sin^2 \theta_2 \right]^{1/2} = \left[ 1 - \left( \frac{n_1}{n_2} \right)^2 \sin^2 \theta_1 \right]^{1/2} \]  

(93)

As a result, we require that

\[ \theta_1 \geq \theta_c \]  

(94)

where the critical angle is defined through

\[ \sin \theta_c = \frac{n_2}{n_1} \]  

(95)

Now, from the geometry of Figure 14.20, we can construct the field distribution of a TE wave in the guide using plane wave superposition. In the slab region \((-d/2 < x < d/2)\), we have

\[ E_{y1s} = E_0 e^{-jk_{1u} x} \pm E_0 e^{-jk_{1d} x} \quad \left( -\frac{d}{2} < x < \frac{d}{2} \right) \]  

(96)

where

\[ k_{1u} = \kappa_1 a_x + \beta a_z \]  

(97)

and

\[ k_{1d} = -\kappa_1 a_x + \beta a_z \]  

(98)

The second term in (96) may either add to or subtract from the first term, since either case would result in a symmetric intensity distribution in the \( x \) direction. We expect this because the guide is symmetric. Now, using \( r = xa_x + za_z \), (96) becomes

\[ E_{y1s} = E_0 [e^{j\kappa_1 x} + e^{-j\kappa_1 x}] e^{-j\beta z} = 2E_0 \cos(\kappa_1 x) e^{-j\beta z} \]  

for the choice of the plus sign in (96), and

\[ E_{y1s} = E_0 [e^{j\kappa_1 x} - e^{-j\kappa_1 x}] e^{-j\beta z} = 2jE_0 \sin(\kappa_1 x) e^{-j\beta z} \]  

(99)

(100)

if the minus sign is chosen. Since \( \kappa_1 = n_1 k_0 \cos \theta_1 \), we see that larger values of \( \kappa_1 \) imply smaller values of \( \theta_1 \) at a given frequency. In addition, larger \( \kappa_1 \) values result in a greater number of spatial oscillations of the electric field over the transverse
dimension, as (99) and (100) show. We found similar behavior in the parallel-plate guide. In the slab waveguide, as with the parallel-plate guide, we associate higher-order modes with increasing values of $\kappa_1$.\(^6\)

In the regions above and below the slab, waves propagate according to wavevectors $k_{2u}$ and $k_{2d}$ as shown in Figure 14.20. Above the slab, for example ($x > d/2$), the TE electric field will be of the form

$$E_{y2s} = E_{02} e^{-j k_{2u} x} e^{-j \beta z}$$

(101)

However, $\kappa_2 = n_2 k_0 \cos \theta_2$, where $\cos \theta_2$, given in (93), is imaginary. We may therefore write

$$\kappa_2 = -j \gamma_2$$

(102)

where $\gamma_2$ is real and is given by (using 93)

$$\gamma_2 = j k_2 = j n_2 k_0 \cos \theta_2 = j n_2 k_0 (-j) \left[ \left( \frac{n_1}{n_2} \right)^2 \sin^2 \theta_1 - 1 \right]^{1/2}$$

(103)

Eq. (101) now becomes

$$E_{y2s} = E_{02} e^{-\gamma_2 (x - d/2)} e^{-j \beta z} \quad (x > \frac{d}{2})$$

(104)

where the $x$ variable in (101) has been replaced by $x - (d/2)$ to position the field magnitude, $E_{02}$, at the boundary. Using similar reasoning, the field in the region below the lower interface, where $x$ is negative, and where $k_{2d}$ is involved, will be

$$E_{y2s} = E_{02} e^{\gamma_2 (x + d/2)} e^{-j \beta z} \quad (x < -\frac{d}{2})$$

(105)

The fields expressed in (104) and (105) are those of surface waves. Note that they propagate in the $z$ direction only, according to $e^{-j \beta z}$, but simply reduce in amplitude with increasing $|x|$, according to the $e^{-\gamma_2 (x - d/2)}$ term in (104) and the $e^{\gamma_2 (x + d/2)}$ term in (105). These waves represent a certain fraction of the total power in the mode, and so we see an important fundamental difference between dielectric waveguides and metal waveguides: in the dielectric guide, the fields (and guided power) exist over a cross section that extends beyond the confining boundaries, and in principle they exist over an infinite cross section. In practical situations, the exponential decay of the fields above and below the boundaries is typically sufficient to render the fields negligible within a few slab thicknesses from each boundary.

\(^6\) It would be appropriate to add the mode number subscript, $m$, to $\kappa_1$, $\kappa_2$, $\beta$, and $\theta_1$, since, as was true with the metal guides, we will obtain discrete values of these quantities. To keep notation simple, the $m$ subscript is suppressed, and we will assume it to be understood. Again, the subscripts 1 and 2 in this section indicate respectively the slab and surrounding regions, and have nothing to do with mode number.
The total electric field distribution is composed of the field in all three regions and is sketched in Figure 14.21 for the first few modes. Within the slab, the field is oscillatory and is of a similar form to that of the parallel-plate waveguide. The difference is that the fields in the slab guide do not reach zero at the boundaries but connect to the evanescent fields above and below the slab. The restriction is that the TE fields on either side of a boundary (being tangent to the interface) must match at the boundary. Specifically,

$$E_{y1s} |_{x = \pm \frac{d}{2}} = E_{y2s} |_{x = \pm \frac{d}{2}}$$

(106)

Applying this condition to (99), (100), (104), and (105) results in the final expressions for the TE electric field in the symmetric slab waveguide, for the cases of even and odd symmetry:

$$E_{se}(\text{even TE}) = \begin{cases} E_{0e} \cos(\kappa_1 x) e^{-j\beta z} & (-\frac{d}{2} < x < \frac{d}{2}) \\ E_{0e} \cos(\kappa_1 \frac{d}{2}) e^{-\gamma_2 (x-d/2)} e^{-j\beta z} & (x > \frac{d}{2}) \\ E_{0e} \cos(\kappa_1 \frac{d}{2}) e^{\gamma_2 (x+d/2)} e^{-j\beta z} & (x < -\frac{d}{2}) \end{cases}$$

(107)

$$E_{so}(\text{odd TE}) = \begin{cases} E_{0o} \sin(\kappa_1 x) e^{-j\beta z} & (-\frac{d}{2} < x < \frac{d}{2}) \\ E_{0o} \sin(\kappa_1 \frac{d}{2}) e^{-\gamma_2 (x-d/2)} e^{-j\beta z} & (x > \frac{d}{2}) \\ -E_{0o} \sin(\kappa_1 \frac{d}{2}) e^{\gamma_2 (x+d/2)} e^{-j\beta z} & (x < -\frac{d}{2}) \end{cases}$$

(108)

Solution of the wave equation yields (as it must) results identical to these. The reader is referred to References 2 and 3 for the details. The magnetic field for the TE modes will consist of $x$ and $z$ components, as was true for the parallel-plate guide. Finally, the TM mode fields will be nearly the same in form as those of TE modes, but with a simple rotation in polarization of the plane wave components by 90°. Thus, in TM modes, $H_x$ will result, and it will have the same form as $E_y$ for TE, as presented in (107) and (108).
Apart from the differences in the field structures, the dielectric slab waveguide operates in a manner that is qualitatively similar to the parallel-plate guide. Again, a finite number of discrete modes will be allowed at a given frequency, and this number increases as frequency increases. Higher-order modes are characterized by successively smaller values of $\theta_1$.

An important difference in the slab guide occurs at cutoff for any mode. We know that $\theta = 0$ at cutoff in the metal guides. In the dielectric guide at cutoff, the wave angle, $\theta_1$, is equal to the critical angle, $\theta_c$. Then, as the frequency of a given mode is raised, its $\theta_1$ value increases beyond $\theta_c$ in order to maintain transverse resonance, while maintaining the same number of field oscillations in the transverse plane.

As wave angle increases, however, the character of the evanescent fields changes significantly. This can be understood by considering the wave angle dependence on evanescent decay coefficient, $\gamma_2$, as given by (103). Note in that equation that as $\theta_1$ increases (as frequency goes up), $\gamma_2$ also increases, leading to a more rapid falloff of the fields with increasing distance above and below the slab. The mode therefore becomes more tightly confined to the slab as frequency is raised. Also, at a given frequency, lower-order modes, having smaller wave angles, will have lower values of $\gamma_2$ as (103) indicates. Consequently, when considering several modes propagating together at a single frequency, the higher-order modes will carry a greater percentage of their power in the upper and lower regions surrounding the slab than will modes of lower order.

One can determine the conditions under which modes will propagate by using the transverse resonance condition, as we did with the parallel-plate guide. We perform the transverse round trip analysis in the slab region in the same manner that was done in Section 14.3, and obtain an equation similar to (37):

$$\kappa_1 d + \phi_{TE} + \kappa_1 d + \phi_{TM} = 2m\pi$$

(109)

for TE waves and

$$\kappa_1 d + \phi_{TM} + \kappa_1 d + \phi_{TM} = 2m\pi$$

(110)

for the TM case. Eqs. (109) and (110) are called the *eigenvalue equations* for the symmetric dielectric slab waveguide. The phase shifts on reflection, $\phi_{TE}$ and $\phi_{TM}$, are the phases of the reflection coefficients, $\Gamma_s$ and $\Gamma_p$, given in (89) and (90). These are readily found, but they turn out to be functions of $\theta_1$. As we know, $\kappa_1$ also depends on $\theta_1$, but in a different way than $\phi_{TE}$ and $\phi_{TM}$. Consequently, (109) and (110) are *transcendental* in $\theta_1$, and they cannot be solved in closed form. Instead, numerical or graphical methods must be used (see References 4 or 5). Emerging from this solution process, however, is a fairly simple cutoff condition for any TE or TM mode:

$$k_0 d \sqrt{n_1^2 - n_2^2} \geq (m - 1)\pi \quad (m = 1, 2, 3, \ldots)$$

(111)

For mode $m$ to propagate, (111) must hold. The physical interpretation of the mode number $m$ is again the number of half-cycles of the electric field (for TE modes)
or magnetic field (for TM modes) that occur over the transverse dimension. The lowest-order mode \((m = 1)\) is seen to have no cutoff—it will propagate from zero frequency on up. We will thus achieve single-mode operation (actually a single pair of TE and TM modes) if we can assure that the \(m = 2\) modes are below cutoff. Using (111), our single-mode condition will thus be:

\[
k_0 d \sqrt{n_1^2 - n_2^2} < \pi
\]

Using \(k_0 = 2\pi / \lambda\), the wavelength range over which single-mode operation occurs is

\[
\lambda > 2d \sqrt{n_1^2 - n_2^2}
\]

A symmetric dielectric slab waveguide is to guide light at wavelength \(\lambda = 1.30 \, \mu m\). The slab thickness is to be \(d = 5.00 \, \mu m\), and the refractive index of the surrounding material is \(n_2 = 1.450\). Determine the maximum allowable refractive index of the slab material that will allow single TE and TM mode operation.

**Solution.** Eq. (113) can be rewritten in the form,

\[
n_1 < \sqrt{\left(\frac{\lambda}{2d}\right)^2 + n_2^2}
\]

So

\[
n_1 < \sqrt{\left(\frac{1.30}{2(5.00)}\right)^2 + (1.450)^2} = 1.456
\]

Clearly, fabrication tolerances are very exacting when constructing dielectric guides for single-mode operation!

**D14.11.** A 0.5 mm thick slab of glass \((n_1 = 1.45)\) is surrounded by air \((n_2 = 1)\). The slab guides infrared light at wavelength \(\lambda = 1.0 \, \mu m\). How many TE and TM modes will propagate?

**Ans.** 2102

### 14.7 OPTICAL FIBER

Optical fiber works on the same principle as the dielectric slab waveguide, except of course for the round cross section. A step index fiber is shown in Figure 14.10, in which a high index core of radius \(a\) is surrounded by a lower-index cladding of radius \(b\). Light is confined to the core through the mechanism of total reflection, but again some fraction of the power resides in the cladding as well. As we found in the slab guide, the cladding power again moves in toward the core as frequency is raised.
Additionally, as is true in the slab waveguide, the fiber supports a mode that has no cutoff.

Analysis of the optical fiber is complicated. This is mainly because of the round cross section, along with the fact that it is generally a three-dimensional problem; the slab waveguide had only two dimensions to be concerned about. It is possible to analyze the fiber using rays within the core that reflect from the cladding boundary as light progresses down the fiber. We did this with the slab guide and obtained results fairly quickly. The method is difficult in fiber, however, because ray paths are complicated. There are two types of rays in the core: (1) those that pass through the fiber axis (z axis), known as meridional rays, and (2) those that avoid the axis but progress in a spiral-like path as they propagate down the guide. These are known as skew rays; their analysis, although possible, is tedious. Fiber modes are developed that can be associated with the individual ray types, or with combinations thereof, but it is easier to obtain these by solving the wave equation directly. Our purpose in this section is to provide a first exposure to the optical fiber problem (and to avoid an excessively long treatment). To accomplish this, we will solve the simplest case in the quickest way.

The simplest fiber configuration is that of a step index, but with the core and cladding indices of values that are very close, that is \( n_1 \approx n_2 \). This is the weak-guidance condition, whose simplifying effect on the analysis is significant. We already saw how core and cladding indices in the slab waveguide need to be very close in value in order to achieve single-mode or few-mode operation. Fiber manufacturers have taken this result to heart, such that the weak-guidance condition is in fact satisfied by most commercial fibers today. Typical dimensions of a single-mode fiber are between 5 and 10 \( \mu m \) for the core diameter, with the cladding diameter usually 125 \( \mu m \). Refractive index differences between core and cladding are typically a small fraction of a percent.

The main result of the weak-guidance condition is that a set of modes appears in which each mode is linearly polarized. This means that light having x-polarization, for example, will enter the fiber and establish itself in a mode or in a set of modes that preserve the x-polarization. Magnetic field is essentially orthogonal to E, and so it would in that case lie in the y direction. The z components of both fields, although present, are too weak to be of significance; the nearly equal core and cladding indices lead to ray paths that are essentially parallel to the guide axis—deviating only slightly. In fact, we may write for a given mode, \( E_z = \eta H_y \), where \( \eta \) is approximated as the intrinsic impedance of the cladding. Therefore, in the weak-guidance approximation, the fiber mode fields are treated as plane waves (nonuniform, of course). The designation for these modes is \( LP_{\ell m} \), meaning linearly polarized, with integer order parameters \( \ell \) and \( m \). The latter express the numbers of variations over the two dimensions in the circular transverse plane. Specifically, \( \ell \), the azimuthal mode number, is one-half the number of power density maxima (or minima) that occur at a given radius as \( \phi \) varies from 0 to \( 2\pi \). \( m \), the radial mode number, expresses the number of maxima that occur along a radial line (at constant \( \phi \)) that extends from zero to infinity.

Although we may assume a linearly polarized field in a rectangular coordinate system, we are obliged to work in cylindrical coordinates for obvious reasons. In a manner that reminds us of Chapter 7, it is possible to write the x-polarized phasor electric field within a weakly guiding cylindrical fiber as a product of three functions.
each of which varies with one of the coordinate variables, \( \rho, \phi, \) and \( z \):

\[
E_x(\rho, \phi, z) = \sum_i R_i(\rho) \Phi_i(\phi) \exp(-j\beta_i z) \tag{114}
\]

Each term in the summation is an individual mode of the fiber. Note that the \( z \) function is just the propagation term, \( e^{-j\beta_z z} \), since we are assuming an infinitely long lossless fiber.

The wave equation is Eq. (58), which we may write for the assumed \( x \) component of \( E_x \), but in which the Laplacian operator is written in cylindrical coordinates:

\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial^2 E_{ss}}{\partial \rho^2} \right) + \frac{1}{\rho^2} \frac{\partial^2 E_{ss}}{\partial \phi^2} + (k^2 - \beta^2) E_{ss} = 0 \tag{115}
\]

where we recognize that the \( \partial^2 / \partial z^2 \) operation, when applied to (114), leads to a factor of \(-\beta^2\). We now substitute a single term of (114) into (115) [since each term in (114) should alone satisfy the wave equation]. Dropping the subscript \( i \), expanding the radial derivative, and rearranging terms, we obtain:

\[
\frac{\rho^2}{R} \frac{d^2 R}{d\rho^2} + \frac{\rho}{R} \frac{dR}{d\rho} + \rho^2 (k^2 - \beta^2) = -\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} \tag{116}
\]

We note that the left-hand side of (116) varies only with \( \rho \), whereas the right-hand side varies only with \( \phi \). Since the two variables are independent, it must follow that each side of the equation must be equal to a constant. Calling this constant \( \ell^2 \), as shown, we may write separate equations for each side; the variables are now separated:

\[
\frac{d^2 \Phi}{d\phi^2} + \ell^2 \Phi = 0 \tag{117a}
\]

\[
\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left[ k^2 - \beta^2 - \frac{\ell^2}{\rho^2} \right] R = 0 \tag{117b}
\]

The solution of (117a) is of the form of the sine or cosine of \( \phi \):

\[
\Phi(\phi) = \begin{cases} 
\cos(\ell \phi + \alpha) \\
\sin(\ell \phi + \alpha)
\end{cases} \tag{118}
\]

where \( \alpha \) is a constant. The form of (118) dictates that \( \ell \) must be an integer, since the same mode field must occur in the transverse plane as \( \phi \) is changed by \( 2\pi \) radians. Since the fiber is round, the orientation of the \( x \) and \( y \) axes in the transverse plane is immaterial, so we may choose the cosine function and set \( \alpha = 0 \). We will thus use \( \Phi(\phi) = \cos(\ell \phi) \).

The solution of (117b) to obtain the radial function is more complicated. Eq. (117b) is a form of Bessel’s equation, whose solutions are Bessel functions.
of various forms. The key parameter is the function $\beta_i \equiv (k^2 - \beta^2)^{1/2}$, the square of which appears in (117b). Note that $\beta_i$ will differ in the two regions: Within the core ($\rho < a$), $\beta_i = \beta_{i1} = (n_1^2 k_0^2 - \beta^2)^{1/2}$; within the cladding ($\rho > a$), we have $\beta_i = \beta_{i2} = (n_2^2 k_0^2 - \beta^2)^{1/2}$. Depending on the relative magnitudes of $k$ and $\beta$, $\beta_i$ may be real or imaginary. These possibilities lead to two solution forms of (117b):

$$R(\rho) = \begin{cases} A J_\ell(\beta_i \rho) & \beta_i \text{ real} \\ B K_\ell(|\beta_i| \rho) & \beta_i \text{ imaginary} \end{cases}$$  \hspace{1cm} (119)$$

where $A$ and $B$ are constants. $J_\ell(\beta_i \rho)$ is the ordinary Bessel function of the first kind, of order $\ell$ and of argument $\beta_i \rho$. $K_\ell(|\beta_i| \rho)$ is the modified Bessel function of the second kind, of order $\ell$, and having argument $|\beta_i| \rho$. The first two orders of each of these functions are illustrated in Figures 14.22a and b. In our study, it is necessary to know

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**Figure 14.22** (a) Ordinary Bessel functions of the first kind, of orders 0 and 1, and of argument $\beta_i \rho$, where $\beta_i$ is real. (b) Modified Bessel functions of the second kind, of orders 0 and 1, and of argument $|\beta_i| \rho$, where $\beta_i$ is imaginary.
the precise zero crossings of the $J_0$ and $J_1$ functions. Those shown in Figure 14.22a are as follows: For $J_0$, the zeros are 2.405, 5.520, 8.654, 11.792, and 14.931. For $J_1$, the zeros are 0, 3.832, 7.016, 10.173, and 13.324. Other Bessel function types would contribute to the solutions in Eq. (119), but these exhibit nonphysical behavior with radius and are not included.

We next need to determine which of the two solutions is appropriate for each region. Within the core ($\rho < a$) we expect to get an oscillatory solution for the field—much in the same manner as we found in the slab waveguide. Therefore, we assign the ordinary Bessel function solutions to that region by requiring that $\beta_1 = (n_1^2k_0^2 - \beta^2)^{1/2}$ is real. In the cladding ($\rho > a$), we expect surface waves that decrease in amplitude with increasing radius away from the core/cladding boundary. The Bessel $K$ functions provide this behavior and will apply if $\beta_2$ is imaginary. Requiring this, we may therefore write $|\beta_2| = (\beta^2 - n_2^2k_0^2)^{1/2}$. The diminishing field amplitude with increasing radius within the cladding allows us to neglect the effect of the outer cladding boundary (at $\rho = b$), as fields there are presumed too weak for this boundary to have any effect on the mode field.

Since $\beta_1$ and $\beta_2$ are in units of $m^{-1}$, it is convenient to normalize these quantities (while making them dimensionless) by multiplying both by the core radius, $a$. Our new normalized parameters become

\begin{equation}
\begin{align*}
u &\equiv a\beta_1 = a\sqrt{n_1^2k_0^2 - \beta^2} \\
w &\equiv a|\beta_2| = a\sqrt{\beta^2 - n_2^2k_0^2}
\end{align*}
\end{equation}

$u$ and $w$ are in direct analogy with the quantities $\kappa_1d$ and $\kappa_2d$ in the slab waveguide. As in those parameters, $\beta$ is the $z$ component of both $n_1k_0$ and $n_2k_0$ and is the phase constant of the guided mode. $\beta$ must be the same in both regions so that the field boundary conditions will be satisfied at $\rho = a$ for all $z$ and $t$.

We may now construct the total solution for $E_{xs}$ for a single guided mode, using (114) along with (118), (119), (120a), and (120b):

\begin{equation}
E_{xs} = \begin{cases}
E_0J_0(u\rho/a)\cos(\ell\phi)e^{-j\beta z} & \rho \leq a \\
E_0[J_1(u)/K_1(w)]K_1(w\rho/a)\cos(\ell\phi)e^{-j\beta z} & \rho \geq a
\end{cases}
\end{equation}

Note that we have let the coefficient $A$ in (119) equal $E_0$, and $B = E_0[J_1(u)/K_1(w)]$. These choices assure that the expressions for $E_{xs}$ in the two regions become equal at $\rho = a$, a condition approximately true as long as $n_1 \approx n_2$ (the weak-guidance approximation).

Again, the weak-guidance condition also allows the approximation $H \approx E/\eta$, with $\eta$ taken as the intrinsic impedance of the cladding. Having $E_s$ and $H_s$ enables us to find the $LP_{\ell m}$ mode average power density (or light intensity) through

\begin{equation}
|\langle S \rangle| = \frac{1}{2}\text{Re}(E_s \times H_s^*) = \frac{1}{2}\text{Re}(E_{xs}H_{ys}^*) = \frac{1}{2\eta}|E_{xs}|^2
\end{equation}
Using (121) in (122), the mode intensity in W/m² becomes

\[ I_{\ell m} = I_0 J^2_\ell \left( \frac{u \rho}{a} \right) \cos^2(\ell \phi) \quad \rho \leq a \]  

(123a)

\[ I_{\ell m} = I_0 \left( \frac{J_\ell(u)}{K_\ell(w)} \right)^2 K^2_\ell \left( \frac{w \rho}{a} \right) \cos^2(\ell \phi) \quad \rho \geq a \]  

(123b)

where \( I_0 \) is the peak intensity value. The role of the azimuthal mode number \( \ell \), as evident in (123a) and (123b), is to determine the number of intensity variations around the circle, \( 0 < \phi < 2\pi \); it also determines the order of the Bessel functions that are used. The influence of the radial mode number, \( m \), is not immediately apparent in (123a) and (123b). Briefly stated, \( m \) determines the range of allowed values of \( u \) that occur in the Bessel function, \( J(u \rho/a) \). The greater the value of \( m \), the greater the allowed values of \( u \); with larger \( u \), the Bessel function goes through more oscillations over the range \( 0 < \rho < a \), and so more radial intensity variations occur with larger \( m \).

In the slab waveguide, the mode number (also \( m \)) determines the allowed ranges of \( \kappa_1 \). As we saw in Section 14.6, increasing \( \kappa_1 \) at a given frequency means that the slab ray propagates closer to the normal (smaller \( \theta_1 \)), and so more spatial oscillations of the field occur in the transverse direction (larger \( m \)).

The final step in the analysis is to obtain an equation from which values of mode parameters (\( u, w, \) and \( \beta \), for example) can be determined for a given operating frequency and fiber construction. In the slab waveguide, two equations, (109) and (110), were found using transverse resonance arguments, and these were associated with TE and TM waves in the slab. In our fiber, we do not apply transverse resonance directly, but rather implicitly, by requiring that all fields satisfy the boundary conditions at the core/cladding interface, \( \rho = a \).\(^7\) We have already applied conditions on the transverse fields to obtain Eq. (121). The remaining condition is continuity of the \( z \) components of \( \mathbf{E} \) and \( \mathbf{H} \). In the weak-guidance approximation, we have neglected all \( z \) components, but we will consider them now for this last exercise. Using Faraday’s law in point form, continuity of \( H_{z2} \) at \( \rho = a \) is the same as the continuity of the \( z \) component of \( \nabla \times \mathbf{E}_s \), provided that \( \mu = \mu_0 \) (or is the same value) in both regions. Specifically

\[ (\nabla \times \mathbf{E}_s)_z |_{\rho=a} = (\nabla \times \mathbf{E}_s)_z |_{\rho=a} \]  

(124)

The procedure begins by expressing the electric field in (121) in terms of \( \rho \) and \( \phi \) components and then applying (124). This is a lengthy procedure and is left as an

\(^7\) Recall that the equations for reflection coefficient (89) and (90), from which the phase shift on reflection used in transverse resonance is determined, originally came from the application of the field boundary conditions.
exercise (or may be found in Reference 5). The result is the eigenvalue equation for LP modes in the weakly guiding step index fiber:

\[
\frac{J_{\ell-1}(u)}{J_{\ell}(u)} = -\frac{w}{u} \frac{K_{\ell-1}(w)}{K_{\ell}(w)}
\]  

(125)

This equation, like (109) and (110), is transcendental, and it must be solved for \( u \) and \( w \) numerically or graphically. This exercise in all of its aspects is beyond the scope of our treatment. Instead, we will obtain from (125) the conditions for cutoff for a given mode and some properties of the most important mode—that which has no cutoff, and which is therefore the mode that is present in single-mode fiber.

The solution of (125) is facilitated by noting that \( u \) and \( w \) can be combined to give a new parameter that is independent of \( \beta \) and depends only on the fiber construction and on the operating frequency. This new parameter, called the normalized frequency, or \( V \) number, is found using (120a) and (120b):

\[
V \equiv \sqrt{u^2 + w^2} = ak_0\sqrt{n_1^2 - n_2^2}
\]

(126)

We note that an increase in \( V \) is accomplished through an increase in core radius, frequency, or index difference.

The cutoff condition for a given mode can now be found from (125) in conjunction with (126). To do this, we note that cutoff in a dielectric guide means that total reflection at the core/cladding boundary just ceases, and power just begins to propagate radially, away from the core. The effect on the electric field of Eq. (121) is to produce a cladding field that no longer diminishes with increasing radius. This occurs in the modified Bessel function, \( K(w\rho/a) \), when \( w = 0 \). This is our general cutoff condition, which we now apply to (125), whose right-hand side becomes zero when \( w = 0 \). This leads to cutoff values of \( u \) and \( V (u_c \text{ and } V_c) \), and, by (126), \( u_c = V_c \). Eq. (125) at cutoff now becomes:

\[
J_{\ell-1}(V_c) = 0
\]

(127)

Finding the cutoff condition for a given mode is now a matter of finding the appropriate zero of the relevant ordinary Bessel function, as determined by (127). This gives the value of \( V \) at cutoff for that mode.

For example, the lowest-order mode is the simplest in structure; therefore it has no variations in \( \phi \) and one variation (one maximum) in \( \rho \). The designation for this mode is therefore \( \text{LP}_{01} \), and with \( \ell = 0 \), (127) gives the cutoff condition as \( J_{-1}(V_c) = 0 \). Since \( J_{-1} = J_1 \) (true only for the \( J_1 \) Bessel function), we take the first zero of \( J_1 \), which is \( V_c(01) = 0 \). The \( \text{LP}_{01} \) mode therefore has no cutoff and will propagate at the exclusion of all other modes provided \( V \) for the fiber is greater than zero but less than \( V_c \) for the next-higher-order mode. By inspecting Figure 14.22a, we see that the next Bessel function zero is 2.405 (for the \( J_0 \) function). Therefore, \( \ell - 1 = 0 \) in (126), and so \( \ell = 1 \) for the next-higher-order mode. Also, we use the lowest value of \( m \) \( (m = 1) \), and the mode is therefore identified as \( \text{LP}_{11} \). Its cutoff \( V \) is \( V_c(11) = 2.405 \). If \( m = 2 \)
were to be chosen instead, we would obtain the cutoff \( V \) number for the LP\(_{12} \) mode. We use the next zero of the \( J_0 \) function, which is 5.520, or \( V_c(12) = 5.520 \). In this way, the radial mode number, \( m \), numbers the zeros of the Bessel function of order \( \ell - 1 \), taken in order of increasing value.

When we follow the reasoning just described, the condition for single-mode operation in a step index fiber is found to be

\[
V < V_c(11) = 2.405
\]

Then, using (126) along with \( k_0 = 2\pi/\lambda \), we find

\[
\lambda > \lambda_c = \frac{2\pi a}{2.405 \sqrt{n_1^2 - n_2^2}}
\]

as the requirement on free-space wavelength to achieve single-mode operation in a step index fiber. The similarity to the single-mode condition in the slab waveguide [Eq. (113)] is apparent. The cutoff wavelength, \( \lambda_c \), is that for the LP\(_{11} \) mode. Its value is quoted as a specification of most commercial single-mode fiber.

**Example 14.6**

The cutoff wavelength of a step index fiber is quoted as \( \lambda_c = 1.20 \) \( \mu \)m. If the fiber is operated at wavelength \( \lambda = 1.55 \) \( \mu \)m, what is \( V \)?

**Solution.** Using (126) and (129), we find

\[
V = 2.405 \frac{\lambda_c}{\lambda} = 2.405 \left( \frac{1.20}{1.55} \right) = 1.86
\]

The intensity profiles of the first two modes can be found using (123a) and (123b), having determined \( u \) and \( w \) values for each mode from (125). For LP\(_{01} \) we find

\[
I_{01} = \begin{cases} 
I_0 J_0^2(u_{01}\rho/a) & \rho \leq a \\
I_0 \left( \frac{J_0(w_{01})}{K_0(w_{01})} \right)^2 K_0^2(w_{01}\rho/a) & \rho > a
\end{cases}
\]

and for LP\(_{11} \) we find

\[
I_{11} = \begin{cases} 
I_0 J_1^2(u_{11}\rho/a) \cos^2 \phi & \rho \leq a \\
I_0 \left( \frac{J_1(u_{11})}{K_1(w_{11})} \right)^2 K_1^2(w_{11}\rho/a) \cos^2 \phi & \rho > a
\end{cases}
\]

The two intensities for a single \( V \) value are plotted as functions of radius at \( \phi = 0 \) in Figure 14.23. We again note the lower confinement of the higher-order mode to the core, as was true in the slab waveguide.

As \( V \) increases (accomplished by increasing the frequency, for example), existing modes become more tightly confined to the core, while new modes of higher order may begin to propagate. The behavior of the lowest-order mode with changing \( V \) is depicted in Figure 14.24, where we again note that the mode becomes more tightly confined as \( V \) increases. In determining the intensities, Eq. (125) must in general be
solved numerically to obtain $u$ and $w$. Various analytic approximations to the exact numerical solution exist, the best of which is the Rudolf-Neumann formula for the $LP_{01}$ mode, valid over the range $1.3 < V < 3.5$:

$$w_{01} \approx 1.1428V - 0.9960$$ (132)

Having $w_{01}$, $u_{01}$ can be found from (126), knowing $V$.

Another important simplification for the $LP_{01}$ mode is the approximation of its intensity profile by a Gaussian function. An inspection of any of the intensity plots of Figure 14.24 shows a resemblance to a Gaussian, which would be expressed as

$$I_{01} \approx I_0 e^{-2\rho^2/\rho_0^2}$$ (133)

where $\rho_0$, termed the mode field radius, is defined as the radius from the fiber axis at which the mode intensity falls to $1/e^2$ times its on-axis value. This radius depends on frequency, and most generally on $V$. A similar approximation can be made for the fundamental symmetric slab guide mode intensity. In step index fiber, the best fit between the Gaussian approximation and the actual mode intensity as given in (130) is given by the Marcuse formula:

$$\frac{\rho_0}{a} \approx 0.65 + \frac{1.619}{V^{3/2}} + \frac{2.879}{V^6}$$ (134)

The mode field radius (at a quoted wavelength) is another important specification (along with the cutoff wavelength) of commercial single-mode fiber. It is important
to know for several reasons: First, in splicing or connecting two single-mode fibers together, the lowest connection loss will be attained if both fibers have the same mode field radius, and if the fiber axes are precisely aligned. Different radii or displaced axes result in increased loss, but this can be calculated and compared with measurement. Alignment tolerance (allowable deviation from precise axis alignment) is relaxed somewhat if the fibers have larger mode field radii. Second, a smaller mode field radius means that the fiber is less likely to suffer loss as a result of bending. A loosely confined mode tends to radiate away more as the fiber is bent. Finally, mode field radius is directly related to the mode phase constant, $\beta$, since if $\mu$ and $\nu$ are known (found from $\rho_0$), $\beta$ can be found from (120a) or (120b). Therefore, knowledge of how $\beta$ changes with frequency (leading to the quantification of dispersion) can be found by measuring the change in mode field radius with frequency. Again, References 4 and 5 (and references therein) provide more detail.

**D14.12.** For the fiber of Example 14.6, the core radius is given as $a = 5.0 \, \mu m$. Find the mode field radius at wavelengths $(a)$ $1.55 \, \mu m$; $(b)$ $1.30 \, \mu m$.

**Ans.** $6.78 \, \mu m; \ 5.82 \, \mu m$