18

Transmission Lines

18.1 Introduction

We have by now learned what capacitors, resistors, and inductors are from the standpoint of electromagnetic field theory. In circuit theory, we usually assume that these elements are lumped (pointlike) and that they are interconnected by means of wire conductors. The current along a wire conductor is assumed to be the same at all points.

Transmission lines consist most frequently of two conductors (some have more, e.g., a three-phase power line). Examples are a coaxial line, a two-wire line, and a stripline. Transmission lines are rare electromagnetic systems that can also be analyzed by circuit-theory tools, although we need electromagnetic theory for determining the transmission-line parameters (i.e., the circuit elements).

Consider a very long section of a transmission line, such as a coaxial line, with perfect conductors and an imperfect dielectric. Let a dc generator of voltage $V$ be connected at one end of the line and a resistor of resistance $R$ at the other. Is there a current along the line? The answer is, of course, yes. However, because there are stray currents through the imperfect dielectric from one line conductor to the other, the current intensity along the two line conductors is not constant. The largest current will be at the generator end, as at that point all the stray currents add up. At the other end of the line, the current intensity through the conductors is the smallest, equal to $V/R$, as sketched in Fig. 18.1a. Note that if the conductors are perfect, the current intensity in the resistor does not depend on the stray currents.
Figure 18.1 (a) A section of a transmission line with perfect conductors and an imperfect dielectric shows stray currents. (b) A ladderlike circuit-theory approximation of the line in (a).

The conclusion that currents at the generator and load ends are different does not fit into the circuit-theory postulate that the current is the same all along a wire that connects circuit elements. Is it possible nevertheless to use circuit theory to analyze this simple circuit? We can subdivide the line section into short segments and represent it as a ladderlike structure with appropriate resistors connected between the conductors of these short line elements, as in Fig. 18.1b. The accuracy of this approximation will increase with the number of segments. For exact representation we need an infinite number of infinitely small segments, but a large number of segments should also give us an accurate result.

If instead of a dc generator we connect an ac generator, the same effect occurs even if the line dielectric is perfect, for now we have capacitive stray currents between the two conductors. However, now the voltage across the load will also differ from that at the generator, in spite of the line conductors being perfect. This is due to small inductive voltage drops across short segments of the line; we know that a line segment of length $\Delta z$ has an inductance $L' \Delta z$ ($L'$ is the line inductance per unit length). Of course, a real transmission line also has a resistance per unit length (due to imperfect conductors), so in addition we will have a resistive voltage drop across segments of the line.

Shown in Table 18.1 are the parameters $C'$, $G'$, $L'$, and $R'$ of the three mentioned transmission-line types. Note that most frequently $\mu = \mu_0$, that the conductivity of the conductors is approximately $\sigma_c \simeq 56 \times 10^6$ S/m (copper), that the relative permittivity of the dielectric is usually 1.0 (air) or 2.1 to 4.0 (most other dielectrics, although dielectrics with considerably higher relative permittivity are also used), and that the conductivity of the dielectrics other than air is on the order of $10^{-12}$ S/m.

Thus if we wish to analyze any transmission line with ac excitation by circuit-theory concepts, we need to represent it as a series connection of many small cells containing series inductors and resistors, and parallel capacitors and resistors, as in Fig. 18.2. Such circuits are said to have distributed parameters. If losses in a line are very small, the line is referred to as a lossless line. Although all lines have losses, they can frequently be neglected, so analysis of lossless lines is of considerable practical interest.
TABLE 18.1 Parameters of Some Transmission Lines at High Frequencies

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Coaxial line</th>
<th>Two-wire line ($d \gg 2a$)</th>
<th>Strip line ($b \gg a$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C'$</td>
<td>$\frac{2\pi \epsilon}{\ln b/a}$</td>
<td>$\frac{\pi \epsilon}{\ln d/a}$</td>
<td>$\frac{b}{\epsilon a}$</td>
</tr>
<tr>
<td>$G'$</td>
<td>$\frac{\sigma_d}{\epsilon} C'$</td>
<td>$\frac{\sigma_d}{\epsilon} C'$</td>
<td>$\frac{\sigma_d}{\epsilon} C'$</td>
</tr>
<tr>
<td>$L_{ext}'$</td>
<td>$\frac{\mu}{2\pi} \ln \frac{b}{a}$</td>
<td>$\frac{\mu}{\pi} \ln \frac{d}{a}$</td>
<td>$\frac{a}{\mu b}$</td>
</tr>
<tr>
<td>$R'$</td>
<td>$\frac{R_s}{\pi a} \left( \frac{1}{a} + \frac{1}{b} \right)$</td>
<td>$\frac{R_s}{\pi a}$</td>
<td>$\frac{2R_s}{b}$</td>
</tr>
</tbody>
</table>

$L_{int}' = R'/\omega$ in all three cases, and $R_s = \sqrt{\omega \mu/2\sigma_c}$ (see Examples 21.7 and 21.9 for proof); $\sigma_c$ is the conductivity of the line conductors. Proofs in Ex 20.4–20.6.

![Circuit-diagram](image)

**Figure 18.2** Circuit-theory approximation of a transmission line with losses

We will develop first the theory for the analysis of lossless lines and then introduce losses in a simple manner. The analysis will show that the time-varying voltage and current along the line vary continuously and that these variations propagate along the line. These are known as **voltage and current waves**. The analysis will also show that the voltages, currents, and the ratio of voltage and current at a point along the line depend on what load is connected at the end of the line. Typically, we wish to efficiently deliver power from a generator, through a line, to the load.

Questions and problems: P18.1 and P18.2

### 18.2 Analysis of Lossless Transmission Lines

The circuit-theory approximations of short and long sections of a lossless transmission line are sketched in Figs. 18.3a and b. Consider the three short sections in
Figure 18.3 (a) A very short piece of a lossless transmission line of length Δz can be represented as a circuit consisting of a series inductor and a shunt (parallel) capacitor. (b) A longer piece of the line can be represented as many cascaded short sections.

Fig. 18.3b, labeled \((n-1), n, \) and \((n+1)\). Let us apply Kirchhoff’s voltage and current laws. The voltage across the \(n\)th inductor and the current through the \(n\)th capacitor are

\[
\Delta L \frac{di_n}{dt} = v_n - v_{n+1} \quad \text{and} \quad \Delta C \frac{dv_n}{dt} = i_{n-1} - i_n. \tag{18.1}
\]

Dividing both equations by \(Δz\) and noting that

\[
\frac{ΔL}{Δz} = L' \quad \text{and} \quad \frac{ΔC}{Δz} = C', \tag{18.2}
\]

we can rewrite Eqs. (18.1) as follows:

\[
L' \frac{di_n}{dt} = \frac{v_{n+1} - v_n}{Δz} \quad \text{and} \quad C' \frac{dv_n}{dt} = \frac{i_n - i_{n-1}}{Δz}. \tag{18.3}
\]

As \(Δz\) approaches zero, the right-hand sides become derivatives with respect to the coordinate \(z\) (note that the left-hand sides are true derivatives with respect to time), and Eqs. (18.3) become

\[
\frac{∂v(t, z)}{∂z} = -L' \frac{∂i(t, z)}{∂t} \quad \text{and} \quad \frac{∂i(t, z)}{∂z} = -C' \frac{∂v(t, z)}{∂t}. \tag{18.4}
\]

(Transmission-line equations, or telegraphers’ equations, for lossless lines)

Partial derivatives need to be used because \(u(z, t)\) and \(i(z, t)\) are functions of time, \(t\), and distance along the line, \(z\). It is clear that if voltage and current vary in time, they also vary along the line. Equations (18.4) are called the transmission-line equations or the telegraphers’ equations. These two equations are coupled differential equations in two unknowns, \(i\) and \(v\).

It is easy to obtain instead equations with only voltage or only current. To that aim, take the derivative with respect to \(z\) (\(∂/∂z\)) of the first equation and the time derivative (\(∂/∂t\)) of the second equation and eliminate the current (or voltage) by
substitution. Following this procedure, we obtain

\[
\frac{\partial^2 v(t, z)}{\partial t^2} - \frac{1}{L'C'} \frac{\partial^2 v(t, z)}{\partial z^2} = 0 \quad \frac{\partial^2 i(t, z)}{\partial t^2} - \frac{1}{L'C'} \frac{\partial^2 i(t, z)}{\partial z^2} = 0. \tag{18.5}
\]

(Wave equations for voltage and current along lossless transmission lines)

These equations are known as the wave equations. They describe the variation of voltage and current along a line and in time. The same or similar type of equation can be used to describe the electric and magnetic fields in a radio wave or optical ray, sound waves in acoustics, etc. We will later derive the same equation for the electric and magnetic field strength vectors, \( E \) and \( H \), instead of voltages and currents.

### 18.2.1 FORWARD AND BACKWARD VOLTAGE WAVES IN THE TIME DOMAIN

Consider the voltage wave equation. It is not difficult to show (see Example 18.1) that its solution is

\[
v(t, z) = V_+ f(t - z/c) + V_- g(t + z/c) \quad (V), \tag{18.6}
\]

(Forward and backward voltage wave on a transmission line)

where \( V_+ \) and \( V_- \) are constants, \( f \) and \( g \) are arbitrary functions of the indicated arguments, and

\[
c = \frac{1}{\sqrt{L'C'}} \quad (m/s). \tag{18.7}
\]

(Velocity of a wave propagating along a transmission line)

The physical meaning of the solution in Eq. (18.6) is as follows. Consider the function \( f(t, z) = f(t - z/c) \) (the constant \( V_+ \) is irrelevant). Let the function at \( t = 0 \) be as \( f(0, z) \) in Fig. 18.4. At a somewhat later instant, say \( t = \Delta t \), the difference \( t - z/c \) will have the same value as for \( t = 0 \) if we consider a point \( z + c \Delta t = z + \Delta z \) instead of point \( z \). This means that the bell-shaped voltage pulse \( f(0, z) \) will have exactly the same form, but will be moved by \( \Delta z = c\Delta t \), as indicated by the pulse labeled \( f(\Delta t, z + c \Delta t) \) in Fig. 18.4. Because \( \Delta t \) is arbitrary, this means that the voltage pulse moves from left to right in Fig. 18.4 (i.e., in the direction of the \( z \) axis) with a velocity \( c = 1/\sqrt{L'C'} \). The wave moving in the \( +z \) direction is the forward traveling (voltage) wave or the incident (voltage) wave.

It is simple to conclude in the same way that the function \( g(t + z/c) \) represents a voltage wave propagating in the \( -z \) direction. Such a wave is the backward traveling (voltage) wave or the reflected (voltage) wave.
Figure 18.4 A voltage wave moves unchanged in shape, with constant velocity $c$, along a lossless transmission line.

What is the velocity of propagation of voltage waves along a typical cable? For a 50-$\Omega$ coaxial cable with a typical dielectric, $C' \simeq 1$ pF/cm, and $L' \simeq 2.5$ nH/cm. The velocity $c$ in Eq. (18.7) for these $C'$ and $L'$ is about two-thirds of the speed of light in air (i.e., about $2 \times 10^8$ m/s).

**Example 18.1—Proof that $f(t - z/c)$ and $g(t + z/c)$ are solutions of the wave equation.**

The proof is simple if we recall the rules for finding the derivatives of a function of several variables. Suppose we have a function $f(x)$, where $x = x(t, z)$ is an arbitrary function of two independent variables, $t$ and $z$. The partial derivative of $f(x)$ with respect to $t$, for example, is obtained using the chain rule as follows:

$$\frac{\partial f(x)}{\partial t} = \frac{\partial f(x)}{\partial x} \frac{\partial x(t, z)}{\partial t}.$$

The second partial derivative is obtained in an analogous manner.

Let $x(t, z) = (t \pm z/c)$. Then

$$\frac{\partial f(x)}{\partial t} = \frac{\partial f(x)}{\partial x} \frac{\partial (t \pm z/c)}{\partial t} = \frac{\partial f(x)}{\partial x}, \quad (18.8a)$$

because $\partial (t \pm z/c)/\partial t = 1$. Hence also

$$\frac{\partial^2 f(x)}{\partial t^2} = \frac{\partial^2 f(x)}{\partial x^2}.$$

The derivative with respect to $z$ is somewhat different because $z$ is multiplied by a constant, $\pm 1/c$:

$$\frac{\partial f(x)}{\partial z} = \frac{\partial f(x)}{\partial x} \frac{\partial (t \pm z/c)}{\partial z} = \pm \frac{1}{c} \frac{\partial f(x)}{\partial x}, \quad (18.8b)$$

and

$$\frac{\partial^2 f(x)}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 f(x)}{\partial x^2}.$$
Substituting the second derivatives with respect to \( t \) and \( z \) into the wave equation (18.5), we see that it is indeed satisfied for any function \( f(t \pm z/c) \).

Note that according to Eqs. (18.8a) and (18.8b),

\[
\frac{\partial f(x)}{\partial z} = \pm \frac{1}{c} \frac{\partial f(x)}{\partial x} = \pm \frac{1}{c} \frac{\partial f(x)}{\partial t}.
\]

(18.9)

**18.2.2 FORWARD AND BACKWARD VOLTAGE WAVES IN THE COMPLEX (FREQUENCY) DOMAIN**

Here we deal mostly with sinusoidally time-varying voltages and currents and linear materials, which means that we can use phasor (complex) notation. Both voltage and current have an assumed exponential time variation,

\[
v, i \propto e^{j\omega t} \quad j = \sqrt{-1} \quad \text{(the imaginary unit)},
\]

(18.10)

but these exponentials cancel out when we write equations involving \( V \) and \( I \) in phasor form. We use capital letters for rms (root mean square) complex quantities. The derivative with respect to time becomes just a multiplication with \( j\omega \). Consequently, we can write the transmission-line equations (18.4) for sinusoidal time variation as

\[
\frac{dV(z)}{dz} = -j\omega L' I(z) \quad \text{and} \quad \frac{dI(z)}{dz} = -j\omega C' V(z).
\]

(18.11)

*Lossless-transmission-line equations in phasor (complex) form*

Eliminating the current from these equations results in

\[
\frac{d^2V(z)}{dz^2} = -\omega^2 L' C' V(z) = (j\omega \sqrt{L'C'})^2 V(z) = (j\beta)^2 V(z).
\]

(18.12)

*Voltage wave equation along lossless transmission lines in phasor (complex) form*

The solution to this second-order differential equation is of the form

\[
V(z) = V_+ e^{-j\beta z} + V_- e^{+j\beta z} \quad \text{(V)},
\]

(18.13)

*(Total voltage wave in phasor notation)*

where

\[
\beta = \omega \sqrt{L'C'} = \frac{\omega}{c} \quad \text{(1/m)}.
\]

(18.14)

*Definition of phase constant*
The constant $\beta$ is known as the phase constant (or phase coefficient) because it determines the phase of the voltage at a distance $z$ from the origin ($z = 0$). Comparing Eq. (18.13) with Eq. (18.6), it can be inferred that $V_+ e^{-j\beta z}$ is the complex representation of a forward traveling wave and $V_- e^{+j\beta z}$ that of a backward traveling wave.

### 18.2.3 WAVELENGTH ALONG TRANSMISSION LINES

Consider the expression for the forward traveling cosine voltage wave in the time domain,

$$v_+(t, z) = V_+ \sqrt{2} \cos(\omega t - \beta z).$$

The argument of the cosine function remains the same if any multiple of $2\pi$ is added to it, or by moving by $\beta \Delta z = n \cdot 2\pi, \ n = \pm 1, \pm 2, \ldots$ along the line. The smallest distance $\Delta z = \lambda$ for which this happens is obtained from the equation $\beta \lambda = 2\pi$, from which we derive

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{(\omega/c)} = \frac{c}{f} \quad \text{(m).}$$

(Definition of wavelength of sinusoidal waves)

This distance, $\lambda$, is known as the wavelength of the sinusoidal wave.

Note again that in complex (phasor) notation, the forward traveling wave has a minus sign in the exponential. This means that for a fixed moment in time the phase of the wave lags along the $z$ direction. (Because the wave is propagating in that direction, this must be the case.)

### 18.2.4 CURRENT WAVES IN THE COMPLEX (PHASOR) DOMAIN, AND THE CHARACTERISTIC IMPEDANCE

Expressions analogous to those for the voltage wave can be obtained for the current wave along the line. From Eqs. (18.4), (18.6), and (18.9), we find

$$\frac{\partial i(t, z)}{\partial t} = -\frac{1}{L'} \frac{\partial v(t, z)}{\partial z} = \frac{1}{cL'} \left[ V_+ \frac{\partial f(t - z/c)}{\partial t} - V_- \frac{\partial g(t + z/c)}{\partial t} \right].$$

This equation can be integrated directly with respect to time. Assuming zero dc components of voltages and currents and having in mind Eq. (18.7), the integration results in

$$i(t, z) = \frac{V_+}{\sqrt{L'/C'}} f(t - z/c) - \frac{V_-}{\sqrt{L'/C'}} g(t + z/c) \quad \text{(A).}$$

(Forward and backward current waves along transmission lines)

In phasor notation this equation becomes
\[ I(z) = \frac{V_+}{Z_0} e^{-j\beta z} - \frac{V_-}{Z_0} e^{+j\beta z} \quad (\Lambda), \]  
(Forward and backward current waves in phasor notation)  

where

\[ Z_0 = \sqrt{\frac{L'}{C'}} \quad (\Omega). \]  
(Characteristic impedance of lossless line)

\[ Z_0 \] is called the characteristic impedance of the lossless transmission line. Like \( L' \) and \( C' \), it depends only on how the line is built (its dimensions and the materials used in it).

**Example 18.2—Numerical values of \( c \) and \( Z_0 \) for some lossless transmission lines.** Because for lossless lines \( L_{\text{int}}' = 0 \), \( R' = 0 \), and \( G' = 0 \), for the three lines given in Table 18.1 the velocity of propagation, \( c \), becomes

\[ c = \frac{1}{\sqrt{L' C'}} = \frac{1}{\sqrt{\varepsilon \mu}} \quad \text{(for the three lines in Table 18.1)}. \]

It can be shown that this simple relation is valid not only for the three lines considered but for all lossless transmission lines with a homogeneous dielectric.

In particular, if the dielectric in the line is air we have

\[ c_0 = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = \frac{1}{\sqrt{8.854 \times 10^{-12} \times 4\pi \times 10^{-7}}} \approx 3 \times 10^8 \text{ m/s}, \]

i.e., the velocity of propagation of voltage and current waves along air lines equals the velocity of light in a vacuum. This is a conclusion to be remembered. Note that it is valid only for lines with air dielectric. Because for any dielectric \( \varepsilon > \varepsilon_0 \), the propagation velocity along lines with dielectric other than air is always less than the velocity of light in a vacuum.

The characteristic impedance, \( Z_0 \), is different for the three lines. Let us write the explicit expression for the coaxial line:

\[ Z_0 = \sqrt{\frac{L'}{C'}} = \frac{1}{2\pi} \sqrt{\frac{\mu}{\varepsilon}} \ln \frac{b}{a} \quad \text{(lossless coaxial line)}. \]

For example, if \( b/a = e = 2.71828 \) and the dielectric is air, we obtain that \( Z_0 \approx 60 \Omega \). Characteristic impedance of commercial coaxial lines (for which \( \varepsilon > \varepsilon_0 \)) ranges from about 50 \( \Omega \) to about 90 \( \Omega \).
Generally, we have both a forward and a backward wave on the line. To calculate the ratio \( \frac{v(t, z)}{i(t, z)} \) at any point along the line and at any instant, the complete expressions for \( v(t, z) \) and \( i(t, z) \) in Eqs. (18.6) and (18.16) must be used. If only a forward wave exists along a line, the ratio of the forward voltage and current waves is found from Eqs. (18.6) and (18.16) to be

\[
\frac{v(t, z)}{i(t, z)} = Z_0. \tag{18.19}
\]

(Only a forward wave along the line)

This means that if only a forward wave exists along the line, the ratio of voltage and current at any point along the line and at any instant of time is the same, equal to \( Z_0 \).

If only the backward wave propagates along the line, Eqs. (18.6) and (18.16) yield

\[
\frac{v(t, z)}{i(t, z)} = -Z_0. \tag{18.20}
\]

(Only a backward wave along the line)

These two equations have a simple physical meaning. Consider Fig. 18.5a and assume reference directions of voltage \( v_+ \) and current \( i_+ \) as indicated. If only a forward wave exists, the generator is at the left in Fig. 18.5a, and the line to the right is such that no backward (reflected) wave is created. This will be the case if the line is infinitely long to the right. This means that a generator connected to the input terminals of an infinitely long lossless line will see the line as a resistor of resistance equal to \( Z_0 \).

The last conclusion enables us to understand an extremely important fact: because an infinite section of any lossless line with respect to its input terminals behaves as a resistor of resistance \( Z_0 \), we can eliminate the reflected wave on a line of any length by terminating the line in its characteristic impedance. If this is done, we say that the line is matched.

Why do we have the minus sign in Eq. (18.20)? Note that the reference directions of voltage and current have been adopted as indicated in Figs. 18.5a and b (the reference conductor for voltage is designated by a "+" sign and the reference direc-

![Diagram](image)

Figure 18.5 (a) Forward and (b) backward voltage and current waves in a transmission line. Note that the adopted reference directions of voltage and current are the same in both cases.
tion for current by an arrow). If the backward wave alone propagates along the line, the generator must be at the right end of the line, feeding an infinite line extending to the left. In Fig. 18.5b the same reference directions are adopted for voltage and current as in Fig. 18.5a; therefore one of these quantities for a reflected wave must be negative so that the power flow is from right to left. Because we retained the same sign for the backward voltage wave, the current wave must change sign with respect to the forward wave. The meaning of the negative sign is exactly the same as in circuit theory: the current is in the direction opposite to the reference direction.

Assume that there is a backward (reflected) wave along the line. Let the generator be connected at the line end toward which the backward wave is propagating (to the left in Fig. 18.5b). When the backward wave reaches the generator, will it produce a backward-backward (i.e., a new forward) wave? The answer is evident: such a wave will be produced unless the internal resistance of the generator equals the line characteristic impedance, \( Z_0 \). For this reason generators (or equivalent Thévenin’s generators), if possible, are made to have this internal resistance, usually 50 \( \Omega \). In what follows, we assume that generators driving transmission lines satisfy this condition, i.e., that they are matched to the line.

Questions and problems: Q18.1 to Q18.11, P18.3 to P18.7

18.3 Analysis of Terminated Lossless Transmission Lines in Frequency Domain

The excitation of transmission lines can have any time variation. Frequently the excitation is sinusoidal or nearly sinusoidal. In this and the next section we restrict our attention to sinusoidal voltages and currents along transmission lines and use the phasor (complex) notation. This section is devoted to lossless lines, and the next to lines with losses.

In reality, a line may be terminated in any load, which is not necessarily the same as its characteristic impedance, as shown in Fig. 18.6. We now know that a forward (or incident) wave travels from the generator to the right in Fig. 18.6. When

![Figure 18.6](image)

Figure 18.6 A transmission line of characteristic impedance \( Z_0 \) terminated in a load \( Z_L \) at a distance \( z_L \) from the generator of internal impedance \( Z_0 \). The coordinate origin, \( z = 0 \), is adopted at the position of the load.
it reaches the load, some of the power is absorbed, and some is reflected, giving rise to a backward wave. The line being linear (the fundamental transmission-line equations are linear), the amplitude of the reflected wave is proportional to that of the incident wave. Because we normally assume that the generator is matched to the line, when the reflected wave reaches the generator it is absorbed in its internal impedance.

The coordinate origin, $z = 0$, can be anywhere along the line. In the analysis of transmission lines, we are mostly concerned with the load because this is where we wish the generator power to be delivered. Therefore, it is convenient to shift the origin from the generator to the load, as in Fig. 18.6.

### 18.3.1 THE REFLECTION AND TRANSMISSION COEFFICIENTS

The (voltage) reflection coefficient is defined as the ratio of the complex amplitudes (or rms values) of the reflected and incident voltage waves at the load. If $z = 0$ is at the load, as in Fig. 18.6, the reflection coefficient is given by

$$\rho = \frac{V_-}{V_+} \quad \text{(dimensionless)}. \quad \text{(18.21)}$$

**(Definition of the reflection coefficient)**

With this definition, the phasor voltage and current along the line in Eqs. (18.13) and (18.17) can be written in the form

$$V(z) = V_+ e^{-j\beta z} (1 + \rho e^{2j\beta z}) \quad \text{(V)}, \quad \text{(18.22a)}$$

$$I(z) = \frac{V_+}{Z_0} e^{-j\beta z} (1 - \rho e^{2j\beta z}) \quad \text{(A)}. \quad \text{(18.22b)}$$

**(Total voltage and current along a transmission line in terms of the reflection coefficient)**

When a generator is connected at one end of a line and a load at the other end, part of the power is reflected from the load (if the load is not perfectly matched) and part of the power is delivered to the load. Generally the goal is to deliver as much power to the load as possible. A quantity that describes the voltage across the load as a function of the incident voltage is called the transmission coefficient, defined by

$$\tau = \frac{V_{\text{load}}}{V_+} = \frac{V_+ + V_-}{V_+} = 1 + \rho \quad \text{(dimensionless)}. \quad \text{(18.23)}$$

**(Definition of transmission coefficient)**

The magnitude of the voltage reflection coefficient for passive loads is smaller than or equal to unity, whereas the magnitude of the transmission coefficient is smaller than or equal to 2. The following examples illustrate this range of values.
Example 18.3—Reflection and transmission coefficients for shorted, open, and matched transmission lines. Let us look at a few simple and extreme examples of terminations (loads) in Fig. 18.6: (1) an open circuit \((Z_L = \infty)\), (2) a short circuit \((Z_L = 0)\), and (3) a matched load \((Z_L = Z_0)\).

At an open end of a line, no current flows between the two conductors. As the adopted reference directions of forward and backward current waves are the same (Figs. 18.5a and b) the reflected current at that point has to be of the same magnitude as the incident current wave, but of opposite sign. According to Eqs. (18.13) and (18.17), the reflected voltage wave at that point is then equal to the incident wave (note reference directions for the two voltages in Fig. 18.5). Consequently, the voltage reflection coefficient at the load for an open end is \(\rho = 1\). From Eq. (18.22a) it follows that at the open-circuited line end the total voltage is twice the incident voltage, corresponding to a voltage transmission coefficient \(\tau = 2\).

At a short-circuited line end, there is no voltage between the two line conductors, corresponding to a transmission coefficient \(\tau = 0\). Referring to reference directions for voltage in Fig. 18.5, as the total voltage at the end of the line has to be zero, the reflected voltage is the negative of the incident voltage. The (voltage) reflection coefficient for a zero load is therefore \(\rho = -1\). According to Eq. (18.22b), the current at the short-circuited end is twice the current of the incident current wave.

For a matched case (load impedance equal to the line characteristic impedance), if we divide Eq. (18.22a) by Eq. (18.22b) and set \(z = 0\) this ratio is equal to the load impedance, in this case \(Z_0\). So we obtain the following equation:

\[
Z_0 = Z_0 \frac{1 + \rho}{1 - \rho}.
\]

This equation can be satisfied only if \(\rho = 0\). This was to be expected because we know that a matched load absorbs the incident wave completely, corresponding to a transmission coefficient of \(\tau = 1\).

Example 18.4—Time-average power delivered to the load. From circuit theory we know that the time-average power delivered to a load is obtained from the phasor voltage across the load, \(V\), and the phasor current in the load, \(I\), as \(P_{\text{av}} = \text{Re}\{V \cdot I^*\}\), where the asterisk denotes a complex conjugate. The voltage and current across the load are obtained from Eqs. (18.22a) and (18.22b) if we set \(z = 0\). Thus the average power delivered to the load terminating a transmission line is

\[
P_{\text{load av}} = \text{Re}\{V(0)I^*(0)\} = \text{Re}\left\{\frac{|V_+|^2}{Z_0} [1 - \rho^2 + (\rho - \rho^*)]\right\}.
\]

Recall that for a complex number \(a = b + jc\), \(a - a^* = (b + jc) - (b - jc) = j2c\). Therefore \((\rho - \rho^*)\) is purely imaginary, so that

\[
P_{\text{load av}} = \frac{|V_+|^2}{Z_0} (1 - |\rho|^2).
\]

(Average power delivered to a transmission-line load) \hspace{1cm} (18.24)

Note that this is precisely the difference of average power of the incident wave, \(|V_+|^2/Z_0\), and of the reflected wave, \(|V_-|^2/Z_0 = |\rho|^2|V_+|^2/Z_0\).

Usually we wish to express the power delivered to the load in terms of the voltage \(V_g\) of the generator connected at the input transmission-line terminals. As
mentioned, we always assume that the generator is matched to the line, i.e., that its internal impedance is equal to the line characteristic impedance. Because for the incident voltage wave we assume an infinite line, the impedance of the line seen by the incident wave at the generator terminals is simply $Z_0$. Therefore, $V_+ = V_g/2$ for a matched generator.

### 18.3.2 IMPEDANCE OF A TERMINATED TRANSMISSION LINE

Consider again the cross section of the line at $z$ in Fig. 18.6. Looking to the right from points 1 and 2, we have a passive network (containing no generators) with two terminals (points 1 and 2). Considering it as a black box, we define its impedance in the usual manner as the ratio of the voltage between the terminals (which is the total voltage) and the corresponding current (which is the total current). (Note that the adopted reference directions of voltage and current in Fig. 18.6 are precisely as needed for an impedance element to the right of points 1 and 2.) This impedance is a function of $z$. According to Eqs. (18.22a) and (18.22b), we have

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{1 + \rho e^{j2\beta z}}{1 - \rho e^{j2\beta z}}.$$

This is the impedance of the line looking toward the load at a distance $z$ from the coordinate origin (assumed at the load). Due to the adopted coordinate origin, the $z$ coordinate of any point is negative (Fig. 18.7). To avoid the minus sign in the expressions to follow, it is convenient to introduce a new coordinate, $\zeta = -z$ (Fig. 18.7), representing the distance from the load. With this change in variable along the line, the expression for the impedance in the last equation becomes

$$Z(\zeta) = Z_0 \frac{1 + \rho e^{-j2\beta \zeta}}{1 - \rho e^{-j2\beta \zeta}}. \quad (18.25)$$

In particular, for $\zeta = 0$ we have that $Z(0) = Z_L$. Thus

$$Z_L = Z_0 \frac{1 + \rho}{1 - \rho}, \quad (18.26)$$

![Figure 18.7](image.png)  
**Figure 18.7** Coordinate origin, $z = 0$, at the load, and the coordinate $\zeta = -z$.  

which is used for determining $Z_L$ if $\rho$ has been determined experimentally. Solving the last equation, we can obtain expressions for $\rho$ and $\tau$ as a function of the load impedance:

$$\rho = \frac{Z_L - Z_0}{Z_L + Z_0} \quad \text{(dimensionless)}, \quad (18.27a)$$

(Alternative expression for reflection coefficient)

$$\tau = \frac{2Z_L}{Z_L + Z_0} \quad \text{(dimensionless)}, \quad (18.27b)$$

(Alternative expression for transmission coefficient)

so that the reflection and transmission coefficients can be determined knowing only the load impedance and the line characteristic impedance.

Finally, if we substitute $\rho$ from Eq. (18.27a) into Eq. (18.25) and recall that $e^{j\alpha} = \cos \alpha + j \sin \alpha$ (Euler’s formula), the input impedance of a section of line of length $\zeta$ terminated in $Z_L$, Eq. (18.25), after simple manipulations becomes

$$Z(\zeta) = Z_0 \frac{Z_L \cos \beta \zeta + jZ_0 \sin \beta \zeta}{Z_0 \cos \beta \zeta + jZ_L \sin \beta \zeta} = Z_0 \frac{Z_L + jZ_0 \tan \beta \zeta}{Z_0 + jZ_L \tan \beta \zeta} \quad (\Omega). \quad (18.28)$$

(Input impedance of line of length $\zeta$ terminated in impedance $Z_L$)

The important thing to remember is that the characteristic impedance $Z_0$ depends only on the way the line is made. The impedance along the line (looking toward the terminating impedance) is quite different: it depends on both $Z_0$ and the terminating impedance but also on the coordinate along the line.

**Example 18.5—Input impedance of an open line.** Assume that the line is open. This corresponds to $Z_L = \infty$ in Eq. (18.28), so that the input impedance of a section of the line of length $\zeta$ becomes

$$Z(\zeta) = -j \frac{Z_0}{\tan \beta \zeta} = -jZ_0 \csc \beta \zeta.$$

If $\beta \zeta < \pi/2$, that is, if $\zeta < \pi/2 \cdot \lambda/(2\pi) = \lambda/4$, $Z(\zeta)$ is a negative imaginary number. This means that the line behaves as a capacitor. (Note, however, that this line behavior is valid only for a line length less than $\lambda/4$!)

You might recall from Chapter 2 that the parasitic inductance of rf chip capacitors makes these elements look predominantly inductive above a certain frequency (the lead inductance is on the order of 1 nH). At microwave frequencies, short sections of open-ended lines are frequently used to obtain in a simple manner a capacitive reactance of a desired value at a given frequency. Note that this reactance depends on frequency in a different way than in the case of a capacitor (for which $X_C = -1/\omega C$).
As an example, consider a short segment of length 1 cm of a 50-Ω line. Assume that the velocity of wave propagation along the line is 0.67c₀. The reactance of this line segment at \( f = 1000 \) MHz is

\[
Z(1 \text{ cm})_{1000\text{MHz}} = -jZ₀ \cot \left( \frac{2\pi}{c/f} \times \xi \right)
\]

\[
= -j50 \cot \left( \frac{2\pi}{0.67 \times 3 \times 10^9/10^9} \times 0.01 \right) \approx -j320\Omega.
\]

which corresponds to a capacitance of \( 1/(2\pi \times 3 \times 10^9 \times 320) \approx 0.5 \) pF (only at 1000 MHz!). It is suggested as an exercise for the reader to calculate the capacitance of the line between 900 MHz and 1100 MHz.

If \( \lambda/4 < \xi < \lambda/2 \), the line behaves as an inductive element; for a still greater length it behaves again as a capacitive element, and so on. It is left as an exercise for the reader to plot \( Z(\xi) \) as (1) a function of frequency and (2) a function of the length of the line in wavelengths.

**Example 18.6—Input impedance of a shorted line.** Assume now that the line is shorted, i.e., that in Eq. (18.28) \( Z_L = 0 \). The input impedance of a section of the line of length \( \xi \) in this case is

\[
Z(\xi) = +jZ₀ \tan \beta \xi.
\]

If \( \beta \xi < \pi/2 \), that is, if \( \xi < \pi/2 \times \lambda/(2\pi) = \lambda/4 \), \( Z(\xi) \) is a positive imaginary number, i.e., the line behaves as an inductor.

You might also recall from Chapter 2 that an inductor has parasitic capacitance between the windings, and as the frequency increases the element looks more and more like a capacitor. Therefore it is hard to make inductors at microwave frequencies. Shorted sections of transmission lines are used frequently at microwave frequencies to obtain in a simple manner an inductive reactance of desired value. Note, however, that this reactance depends on frequency in a different way from that of an inductor (\( X_L = \omega L \)).

If \( \lambda/4 < \xi < \lambda/2 \), the line behaves as a capacitive element; for a still greater length it behaves again as an inductive element, and so on. It is left as an exercise for the reader to plot \( Z(\xi) \) as (1) a function of frequency and (2) as a function of the length of the line in wavelengths.

**Example 18.7—Quarter-wave transformers.** An interesting and important case is when the length of the transmission line is a quarter of a wavelength. Because then \( \beta \xi = (2\pi/\lambda) \times (\lambda/4) = \pi/2 \), from Eq. (18.28) we obtain

\[
Z \left( \frac{\lambda}{4} \right) = \frac{Z₀}{Z_L}.
\]

The load impedance is transformed from a value \( Z_L \) to a value \( Z₀^2/Z_L \). For example, if \( Z_L \) is a high impedance, \( Z \) will be low and vice versa. Quarter-wavelength transmission-line sections often play the same role at rf and microwave frequencies as impedance transformers at lower frequencies. (At high frequencies, it is difficult to build good transformers due to parasitic capacitances in inductors and also losses in the conductors and cores.)
Quarter-wave transformers are especially used for matching resistive loads. For example, if we want to match a 100-Ω load to a 50-Ω transmission line, we could use a quarter-wavelength section of a line with a characteristic impedance of \( Z_0 = \sqrt{100 \cdot 50} = 70.7 \, \Omega \).

However, unlike in a low-frequency transformer there is a phase lag in the section of the transmission line. This type of impedance transformer does not work for voltage and current transformation. Note also that the quarter-wavelength transformer effect works only in a narrow range of frequencies (it exactly works only at the frequency for which the length of the line is a quarter of a wavelength).

Analogous ideas are used in optics to make antireflection coatings for lenses. We explain this in more detail in later chapters.

**Example 18.8—Thévenin equivalent of an open-ended section of transmission line fed by a generator.** Line terminated in an infinite line of different characteristic impedance. Assume that a line of characteristic impedance \( Z_1 \) (line 1) is terminated in an infinite (or matched) line of characteristic impedance \( Z_2 \) (line 2). We know that line 2 from its input terminals represents a load of resistance \( Z_2 \). So line 1 can be regarded as terminated in a load \( Z_L = Z_2 \). Consequently we know the voltage and current distribution along line 1.

Along line 2 we have only a forward voltage and a forward current wave, propagating along it with a velocity determined by its capacitance and inductance per unit length. For determining these waves we need only determine the voltage at the input terminals of line 2. This can be done very simply by applying Thévenin's theorem.

Line 1 as seen from the input terminals of line 2 represents an equivalent real voltage generator. The voltage of the generator equals the open-circuit voltage at the end of line 1. From Example 18.3 we know that this voltage is twice the incident voltage along line 1, \( V_{Th} = V_+ (1 + \rho) = 2V_+ \). The internal impedance of the Thévenin generator is the impedance of the infinite (or matched) line 1, so \( Z_{Th} = Z_1 \).

Let us summarize this very useful result. The equivalent Thévenin voltage source and impedance for an open-ended section of line of characteristic impedance \( Z_1 \) fed by a generator that gives an incident voltage \( V_+ \) are

\[
Z_{Th} = Z_1 \quad \text{and} \quad V_{Th} = 2V_+.
\]

After we replace line 1 with its Thévenin equivalent, the input voltage of line 2 can be found as in a voltage divider:

\[
V_{2 \, \text{input}} = V_{Th} \frac{Z_2}{Z_1 + Z_2} = 2V_+ \frac{Z_2}{Z_1 + Z_2}.
\]

We know the reflection coefficient for line 1, given in Eq. (18.27a), which in this case becomes

\[
\rho = \frac{Z_2 - Z_1}{Z_2 + Z_1}.
\]

The transmission coefficient, from line 1 to line 2, is given by Eq. (18.27b):

\[
\tau = \frac{V_{2 \, \text{input}}}{V_+} = \frac{2Z_2}{Z_1 + Z_2}.
\]
18.3.3 THE VOLTAGE STANDING-WAVE RATIO (VSWR)

A useful and frequently used concept related to the reflection coefficient is the voltage standing-wave ratio, or VSWR. The VSWR is the ratio of the maximal to minimal voltage along the line. Because $|e^{2i\beta z}| = 1$, according to Eq. (18.22a) the VSWR is given by

$$VSWR = \frac{V(z)_{\text{max}}}{V(z)_{\text{min}}} = \frac{1 + |\rho|}{1 - |\rho|}. \quad (18.30)$$

(Definition of voltage standing-wave ratio, VSWR)

Note that for a matched load, $VSWR = 1$, and for open and for short circuits, $VSWR = \infty$.

Example 18.9—Standing waves on transmission lines. When a line is matched at its end, we know that there is only a forward wave propagating along the line. To visualize such a voltage wave for sinusoidal excitation, imagine a sine function that moves along the line with a velocity $c$.

When the line is not matched there is another sinusoid, usually of smaller amplitude, moving from the load toward the generator (where we assume a matched load, i.e., no more reflected waves) with the same velocity, $c$. So in the general case we have two sine waves of unequal amplitudes moving in opposite directions with the same velocity. The total voltage at any point along the line (and at any moment) is obtained as their sum. Due to their equal velocities, however, there will be fixed minima and maxima of the total wave, as the following example shows.

We know that for a shorted line the voltage reflection coefficient $\rho = -1$ (see Example 18.3). Consequently, according to Eq. (18.22a), the total voltage along the line is of the form

$$V(z) = V_+ e^{-i\beta z} (1 - e^{2i\beta z}) = V_+ (e^{-i\beta z} - e^{i\beta z}) = -j2V_+ \sin(\beta z),$$

because $e^{-ix} - e^{ix} = (\cos x - j\sin x) - (\cos x + j\sin x) = -j2\sin x$.

The instantaneous value of the voltage along the line, $v(t, z)$, is hence obtained as

$$v(t, z) = \text{Re}\{-j2V_+ \sqrt{2} \sin(\beta z)e^{j\omega t}\} = 2V_+ \sqrt{2} \sin(\beta z) \sin \omega t.$$

This voltage does not have any argument of the form $(t \mp z/c)!$ Consequently this is not a forward or a backward traveling voltage wave. Instead, it has zero values at all points where the sine has a zero value, and it oscillates between these fixed, stationary zeros of the total voltage. For this reason, this kind of wave is termed the standing wave. A sketch of the voltage standing wave for a sequence of time instants is shown in Fig. 18.8a. Note that a standing wave in phasor (complex) notation is easily recognized by the absence of the “traveling-wave” factor $e^{\mp i\beta z}$ (or any other coordinate instead of $z$).

The distance along two fixed, zero-voltage points along the line is given by

$$\Delta z_{\text{between two voltage zeros}} = \frac{\pi}{\beta} = \frac{\lambda}{2\pi} = \frac{\lambda}{2}. $$
Figure 18.8 (a) Voltage and (b) current standing wave along a shorted transmission line at indicated time instants (corresponding to the expressions derived in Example 18.9)

The total current along a shorted line has the same property, i.e., it is also a standing wave. From Eq. (18.22b), it is easily found that

\[ i(t, z) = -2 \frac{V_0}{Z_0} \sqrt{2} \cos(\beta z) \cos \omega t. \]

A sketch of the current standing wave for a sequence of time instants is shown in Fig. 18.8b.

It is left as an exercise for the reader to derive the expressions for standing voltage and current waves for an open transmission line.

Thus if we are able to measure the voltage along a transmission line we can easily conclude whether the line is shorted or open. The next example will show how we can measure the impedance of any load terminating a line by measuring, essentially, the VSWR.

**Example 18.10—Measurement of load impedance using a slotted line.** Figure 18.9a shows what is called the slotted coaxial line. Slotted lines may be used for measuring impedances at very high frequencies. To understand how this can be done, let the generator have a fixed but unknown frequency, and let the dielectric in the slotted line of characteristic impedance \( Z_0 \) be air. The coax is rigid and the outer conductor tube has a narrow slot along its length. The slot is made along the current-flow lines in the outer line conductor, so it only slightly affects the distribution of current and voltage in the line. A movable fixture is attached to the tube and contains a pinlike probe that protrudes through the slot and samples the electric field (voltage) inside the cable. Recall that the electric field vector in the cable is radial, so a voltage \( v = \int_{pin}^{} \mathbf{E} \cdot dl \) is induced in the probe. This voltage is converted to a dc voltage by means of a diode detector and gives a measure of the relative electric field along the line. The position of the probe is measured along an arbitrary scale, e.g., like the one in Fig. 18.9b.

Usually the probe cannot slide all the way to the end of the line where the load is attached, and also the connector at the end of the line adds an unknown line length. So the first step in measuring an unknown load is to determine exactly where the line ends. We know that everything along a line is repeated every half wavelength. This means that we can determine the position of the end of the line displaced by an integer number of half wavelengths, so that it falls along our scale.
How is the position of the end of the line determined? The easiest way is to connect a known load to the end, so that a standing wave is set up. If a short (or open) is connected, we know that the minima (maxima) of the standing wave occur at the load and every half wavelength away toward the generator. The wavelength measured along the line is practically equal to the free-space wavelength, \( \lambda_0 = c_0 / f \). Therefore we can measure the frequency of the generator simply by moving the probe back and forth and determining the two successive minima. Usually a short is used because minima are sharper and therefore more precise than maxima. (Sketch the derivative, or slope, of a standing wave to convince yourself.) The standing wave pattern due to a short is sketched in Fig. 18.9b in solid line, and the real and displaced positions of the end of the line are indicated.

After this calibration is performed, the unknown load is connected to the end of the line and the standing wave sampled once more. Again, two successive minima will be at a distance \( \lambda_0 / 2 \) apart, but they are displaced in position from the minima obtained with a short. This is because the phase of the load is different from that of the short. By moving the probe back and forth, we determine the maximum and minimum readings of the indicator, which gives us the voltage standing-wave ratio, VSWR, defined in Eq. (18.30), as sketched in Fig. 18.9b in dashed line. Then we locate as precisely as possible the distance \( \xi_{\min} \) of the first minimum from the minimum obtained with a short, in terms of wavelength. From Eq. (18.22a) we find that the voltage minimum occurs when \( \rho e^{-|\rho|} \) is real and negative, that is, equal to \(-|\rho|\). So the impedance \( Z(\xi) \) given by Eq. (18.28) for \( \xi = \xi_{\min} \) is real, and equal to

\[
Z(\xi_{\min}) = Z_0 \frac{1 - |\rho|}{1 + |\rho|} = \frac{Z_0}{\text{VSWR}}.
\]

As we now know \( Z(\xi_{\min}) \) and \( \xi_{\min} \) (in terms of wavelength), we can evaluate the unknown load impedance \( Z_l \) from Eq. (18.28).

Questions and problems: Q18.12 to Q18.16, P18.8 to P18.27
18.4 Lossy Transmission Lines

We know that a real transmission line has losses in the conductors (due to the finite conductivity of the metal) as well as in the dielectric between the conductors (due principally to the polarization losses in the dielectric). If the line is represented as a series connection of many short cells, these losses can be accounted for by a series and a shunt resistor in every cell, as in Fig. 18.10. The total series impedance per unit length is thus \( R' + j\omega L' \) (instead of \( j\omega L' \) for lossless lines), and the total shunt admittance per unit length is \( G' + j\omega C' \) (instead of \( j\omega C' \)). The phasor equations (18.11) therefore take the form

\[
\frac{dV(z)}{dz} = -(R' + j\omega L')I(z), \quad \text{and} \quad \frac{dI(z)}{dz} = -(G' + j\omega C')V(z). \tag{18.31}
\]

Noting that the lossless-line characteristic impedance in phasor form, \( \sqrt{L'/C'} \), originally was \( \sqrt{j\omega L'/j\omega C'} \), the characteristic impedance of a lossy line is given by

\[
Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}. \tag{18.32}
\]

(Characteristic impedance of a lossy line)

Similarly, the expression \( j\beta = j\omega \sqrt{L'C'} = \sqrt{(j\omega L')(j\omega C')} \) in the exponential terms in Eq. (18.13) now becomes

\[
\gamma = \alpha + j\beta = \sqrt{(R' + j\omega L')(G' + j\omega C')}. \tag{18.33}
\]

(Propagation constant of a lossy line)

The constant \( \gamma \) is known as the propagation constant (or propagation coefficient) of the line, \( \alpha \) as the attenuation constant (coefficient), and \( \beta \), as earlier, the phase constant (coefficient).

Thus, for lossy lines and a forward wave, instead of \( e^{-j\beta z} \) in the expressions for voltages and currents we now have \( e^{-(\alpha+j\beta)z} = e^{-\alpha z}e^{-j\beta z} \). The factor \( e^{-\alpha z} \) means that in addition to traveling in the \( z \) direction, the amplitudes of the forward voltage and current waves also fall off in the direction of propagation. This is called attenuation and is a characteristic of every real transmission line. The phase of the wave is determined by \( \beta \) (phase constant), and its attenuation by \( \alpha \).

\[
\begin{align*}
&\text{(R' + j}\omega L')\Delta z \\
&\text{(G' + j}\omega C')\Delta z \\
&\text{Δz} \\
&\text{→ z}
\end{align*}
\]

Figure 18.10 Schematic of a transmission line with distributed losses included.
Rearranging Eq. (18.33) we obtain
\[
\gamma = \sqrt{j\omega L' j\omega C'} \left( 1 + \frac{R'}{j\omega L'} \right) \left( 1 + \frac{G'}{j\omega C'} \right) = \\
= j\omega \sqrt{L'C'} \sqrt{1 - j \left( \frac{R'}{\omega L'} + \frac{G'}{\omega C'} \right) - \frac{R'G'}{\omega^2 L'C'}}. 
\tag{18.34}
\]

For transmission lines with small losses \((R' \ll \omega L' \text{ and } G' \ll \omega C')\), this can be written in approximate form
\[
\gamma \simeq j\omega \sqrt{L'C'} \sqrt{1 - j \left( \frac{R'}{\omega L'} + \frac{G'}{\omega C'} \right) \simeq j\omega \sqrt{L'C'} \left[ 1 - \frac{j}{2} \left( \frac{R'}{\omega L'} + \frac{G'}{\omega C'} \right) \right]. 
\tag{18.35}
\]

So we find that for transmission lines with small losses
\[
\alpha \simeq \frac{1}{2} \left( R' \sqrt{\frac{C}{L}} + G' \sqrt{\frac{L}{C}} \right) \quad \beta \simeq \omega \sqrt{L'C'}. 
\tag{18.36}
\]

(Attenuation and phase constant for lines with small losses)

If along a transmission line only the forward wave is propagating, both current and voltage along the line have the attenuation factor \(e^{-\alpha z}\). Therefore the average power transmitted at a point \(z\) in the direction of the wave, being the product of phasor rms voltage and conjugate current, is of the form \(P(z) = P(0)e^{-2\alpha z}\).

If a quantity (e.g., voltage) at \(z\) is of amplitude \(V_+e^{-\alpha z}\), at \(z + d\) it is of amplitude \(V_+e^{-\alpha(z+d)}\). The attenuation of the voltage along this line section is frequently expressed as the natural logarithm of the ratio of the voltage amplitude at \(z\) and that at \(z + d\). The unit of this measure of attenuation is termed the neper (Np) [after the Scottish mathematician John Neper (Napier), who at the turn of the 16th century invented the logarithm]:

Attenuation of forward voltage wave in nepers
\[
= \ln \left( \frac{V_+e^{-\alpha z}}{V_+e^{-\alpha(z+d)}} \right) = \ln e^{\alpha d} = \alpha d \quad \text{(Np)} 
\tag{18.37}
\]

The unit of the attenuation constant, \(\alpha\), is thus neper per meter (Np/m).

The attenuation of voltage or current along a line section is more often expressed in terms of decimal logarithm in decibels (dB) (after Alexander Graham Bell, 1847–1922, inventor of the telephone), as

Attenuation of forward voltage wave in decibels
\[
= 20 \log \left( \frac{V_+e^{-\alpha z}}{V_+e^{-\alpha(z+d)}} \right) = 20 \log e^{\alpha d} = (20 \log e) \alpha d \quad \text{(dB)}.
\tag{18.38}
\]

Since \(20 \log e = 8.686\), the attenuation in decibels is 8.686 times the attenuation in nepers, or 1 Np = 8.686 dB.

Questions and problems: Q18.17, P18.28
18.5 Basics of Analysis of Transmission Lines in the Time Domain

For various reasons, cables might have, or develop in use, faults along their length. It is useful to know where, so that they can be quickly repaired without pulling the whole cable out. The instrument used today to find faults in cables is called the time domain reflectometer (TDR). Its operating principle is very simple: the instrument sends a voltage step and waits for the reflected signal. If there is a fault in the cable it will be equivalent to some rapid change in cable properties, and part of the voltage step wave will reflect off the discontinuity. As both the transmitted and reflected waves travel at the same velocity, the distance of the fault from the place where the TDR was connected can be calculated exactly. Not only can we learn where the fault is, but the TDR can also tell us something about the nature of the fault.

So far, we have looked at transmission lines only in the frequency domain (we assumed sinusoidal voltages and currents). Now we will look at what happens when a step function (in time) is launched down a transmission line terminated in a load. To analyze the time-domain response, we first replace the entire line with its Thévenin equivalent with respect to the load, as derived in Example 18.8. We thus obtain a simple circuit with a Thévenin generator connected to a load. Transients in such a circuit can next be analyzed by solving a differential equation, or by the Laplace (or Fourier) transform (the two procedures are basically the same). We will use the latter method, where we multiply the Laplace (or Fourier) transform of the reflection coefficient (i.e., the reflection coefficient in complex form) with the transform of a step function and then transform back to the time domain with the inverse Laplace (or Fourier) transform.

Example 18.11—Reflection from an inductive load. Let us consider reflection from an inductive load (Fig. 18.11a). The transmission line has a characteristic impedance $Z_0$ and the incident voltage wave is $v_i(t)$. The Thévenin equivalent generator and impedance for this line are $Z_{th} = Z_0$ and $v_{th}(t) = 2v_i(t)$ (Fig. 18.11b).

If we now assume that the incident voltage wave is a unit step function starting at $t = 0$, $v_i(t) = 1$, $t > 0$, the Laplace transform is

$$v_+(s) = \frac{1}{s},$$

![Figure 18.11](image)

(a) A transmission line with an incident wave $v_+$, terminating in an inductive load. (b) The lossless transmission line is replaced by its Thévenin equivalent circuit.
and because the impedance of the inductor is

\[ Z = sL, \]

we find that the load voltage is equal to

\[ v(s) = \frac{2L}{sL + Z_0} = \frac{2}{s + Z_0/L}. \]  \hfill (18.39)

We can recognize this as the Laplace transform of a decaying exponential with a time constant \( t_L = L/Z_0 \):

\[ v(t) = 2e^{-t/t_L} \quad t > 0. \]  \hfill (18.40)

Because the voltage of an inductor is \( v(t) = L \frac{di}{dt} \), we can find the current through the inductive load by integrating the voltage:

\[ i(t) = \frac{1}{L} \int_0^t v(t) \, dt = \frac{2}{Z_0} (1 - e^{-t/t_L}) \quad t > 0. \]  \hfill (18.41)

This describes the buildup of current in an inductor through a resistor, which we already understand from circuit theory. The reflected wave is the difference between the transmitted wave and the incident wave:

\[ v_-(t) = v(t) - v_+(t) = 2e^{-t/t_L} - 1 \quad t > 0. \]  \hfill (18.42)

We can see that initially the inductor has no current, and the voltage is just \( v_+ \), so it looks like an open circuit and the reflection coefficient is +1. The current then builds up to the short circuit current (the Norton equivalent current) and the voltage drops to zero, so the inductor appears as a short circuit. The reflected and transmitted waves are shown in Fig. 18.12.

**Example 18.12—Reflection from a short circuit.** As another example, let us look at a transmission line that is shorted at one end. If a voltage source is turned on at the other end, what will the reflected wave look like back at the source? We know that the reflection coef-

![Figure 18.12 The incident, reflected, and total voltages for an inductive load](image)

**Figure 18.12 The incident, reflected, and total voltages for an inductive load**
Figure 18.13 (a) Reflected voltage wave off a short-circuited transmission line with an incident unity step function. (b) A standard TDR display shows the reflected wave added on to the incident step function.

Example 18.13—Reflection from a series RL circuit. A third, slightly more complicated, example is that of a series combination of an inductor $L$ and a resistor $R$. The incident voltage is a step of unit amplitude. The voltage across the inductor is, as before,

$$v_L(t) = 2e^{-t/t_L}, \quad t > 0,$$

(18.43)

where the time constant is now $t_L = L/(R + Z_0)$, because the inductor sees a series connection of the characteristic impedance and the resistive load. The inductor current is

$$i_L(t) = \frac{2}{Z_0 + R}(1 - e^{-t/t_L}), \quad t > 0,$$

(18.44)

and the load voltage becomes

$$v(t) = v_L(t) + Ri_L(t) = 2\left[\frac{R}{R + Z_0} + \frac{Z_0}{R + Z_0}e^{-t/t_L}\right], \quad t > 0.$$

(18.45)

The reflected voltage wave, shown in Fig. 18.14a, is now

$$v_-(t) = v(t) - v_+(t) = \left[\frac{R - Z_0}{R + Z_0} + 2\frac{Z_0}{R + Z_0}e^{-t/t_L}\right], \quad t > 0.$$

(18.46)
Figure 18.14 (a) Reflected voltage wave off a series RL combination with an incident unity step function. (b) A standard TDR instrument display shows the reflected wave added on to the incident step function.

A simpler qualitative analysis can be done by just evaluating the reflected voltage at \( t = 0 \) (the time when the reflected wave gets back to the launching port, for example) and \( t = \infty \), and assuming any transition between these two values to be exponential. In the previously analyzed case of a series RL circuit, at \( t = 0 \) the reflected voltage is \( v_-(0) = +1 \), since the inductor looks like an open circuit initially. On the other hand, as time goes by the current through the inductor builds up, and at \( t = \infty \) the inductor looks like a short, so \( v_-(\infty) = (R - Z_0)/(R + Z_0) \) and is determined by the resistive part of the load. The resulting plot out of a TDR (incident step plus reflected wave) is shown in Fig. 18.14b.

Example 18.14—Measuring the time constant of the reflected wave from complex loads. The most straightforward way to measure the time constant (such as \( t_L \) in the inductor examples) is to measure the time \( t_1 \) needed to complete half of the exponential transition from \( v_-(0) \) to \( v_-(\infty) \). This corresponds to \( t_1 = t_L/0.69 \), where \( t_L \) is the time constant we used for an inductive load, but it also holds for a capacitive load. This procedure is shown qualitatively in Fig. 18.15.

Questions and problems: Q18.18 to Q18.21, P18.29 and P18.30

Figure 18.15 Determining the time constant of an exponential TDR response
18.6 The Graphical Solution of Lossless-Line Problems Using the Smith Chart

Until the advent of digital computers, the solution of transmission-line problems was done most often with the aid of a graphical tool known as the Smith chart (P. H. Smith, "Transmission-line Calculator," *Electronics*, 12, January 1939, p. 29; "An Improved Transmission-line Calculator," *Electronics*, 17, January 1944, p. 130). The Smith chart is a polar plot of the reflection coefficient with some additional details. We restrict our attention to the Smith chart used for solving problems with lossless lines, that is, for $Z_0$ real.

The Smith chart enables us to make a direct determination of the complex reflection coefficient $\rho(0)$ at the load, corresponding to a given load impedance $Z_L$ and the characteristic impedance of the line. Conversely, if $\rho(0)$ is determined experimentally, we can read directly from the chart the load impedance $Z_L$ if $Z_0$ is known. However, the usefulness of the Smith chart far surpasses these two relatively simple tasks, and its use does not seem to decline. Even in the most advanced measurement instruments, such as network analyzers, a Smith chart can be generated on the screen to represent the measurement results, because of the very compact form of such representation. Therefore, we will illustrate the use of a Smith chart with a number of examples. The theoretical basis of the Smith chart is given in most higher-level books on electromagnetics and microwave engineering.

A chart in its usual form, with some additional details whose use will be explained later, is shown in Fig. 18.16. As we have already mentioned, the Smith chart is used for plotting impedances and reflection coefficients. An impedance is plotted on the chart as a *normalized impedance*, defined as

$$z = \frac{Z}{Z_0} = \frac{R + jX}{Z_0} = r + jx \quad \text{(dimensionless).}$$  \hspace{1cm} (18.47)

*Definition of normalized load impedance*

The real part of the impedance, $r$, is defined by the complete circles on the chart. The imaginary part $x$ is defined by the circular arcs. A normalized complex impedance, $z = r + jx$, is defined by the intersection of a circle and an arc. For example, the circle labeled $r = 1$ in Fig. 18.16 intersects the arc labeled $jx = j1$ at the point labeled $z$, which corresponds to an impedance of $Z = z \cdot Z_0 = Z_0(1 + j1)$. If $Z_0 = 50 \, \Omega$, this corresponds to $Z = 50 + j50 \, \Omega$.

The complex reflection coefficient corresponding to $z$ is plotted in polar form, $\rho = |\rho| \angle \phi$, by drawing a straight line from the center of the chart to point $z$. The distance of the point $z$ from the chart center gives $|\rho|$, which can be read off the scale on the horizontal line going through the center of the chart. The angle $\angle \phi$ is read off the (innermost) angular scale on the outer circle of the chart.

The basic properties of the Smith chart are the following:

- All points on the horizontal axis (labeled $r$) correspond to purely real impedances.
Figure 18.16 The Smith chart
• All points on the circle bounding the chart (labeled $x$) correspond to purely imaginary impedances.

• The rightmost point on the chart corresponds to an open circuit (labeled O).

• The leftmost point on the chart corresponds to a short circuit (labeled S).

• The center of the chart has a reflection coefficient equal to zero and corresponds to a matched load (labeled M).

• All points in the lower chart half correspond to loads with a capacitive (negative) reactance.

• All points in the upper chart half correspond to loads with an inductive (positive) reactance.

• The points on the circle labeled $r = 1$ correspond to loads with a real part equal to the adopted normalizing characteristic impedance, $Z_0$ (usually 50 $\Omega$).

• The points on the arc labeled $jx = j1$ correspond to loads with a positive imaginary part equal to the adopted normalizing impedance, $Z_0$ (usually 50 $\Omega$).

• The points on the arc labeled $jx = -j1$ correspond to loads with a negative imaginary part equal to the adopted normalizing impedance, $Z_0$ (usually 50 $\Omega$).

• The length of the straight-line segment between the chart center and a point in the chart represents the magnitude of the reflection coefficient, the points on the boundary circle being of magnitude one. The angle scale on the chart boundary gives the reflection coefficient angle.

• The inside of the Smith chart corresponds to passive impedances (no generators). The magnitude of the reflection coefficient is smaller than or equal to unity inside the chart.

• The outside of the Smith chart (not plotted in Fig. 18.16) corresponds to impedances that give reflection coefficients of magnitudes larger than unity. This means that we can use the chart for active circuits, such as amplifiers and oscillators (i.e., generators). This external part of the chart is sometimes also plotted, and such a chart is referred to as an “extended Smith chart.”

Example 18.15—Determination of the reflection coefficient at the load. The information that can be obtained directly from a Smith chart is the complex reflection coefficient at the load, $\rho = \rho(0)$, corresponding to a certain normalized load impedance $z = r+jx$. For example, for $Z_0 = 50 \Omega$ and $Z_L = (40+j60) \Omega$ we have $z = Z_L/Z_0 = 0.8 + j1.2$, so $r = 0.8$ and $x = +1.2$. From Fig. 18.16, we find that $|\rho| \approx 0.57$, and $\theta_\rho \approx 66^\circ$.

The magnitude of $\rho$ is obtained by first measuring the distance of the point $M$ from the chart center (point $\rho' = \rho'' = 0$), using a compass. Below the chart a linear scale is provided, which we can use to obtain the distance measured by the compass in terms of the chart radius (unit circle in the complex $\rho$ plane). For easy reading of the angle $\theta_\rho$, an angle scale marked “angle of reflection coefficient in degrees” is provided around the main chart. So we only need to draw a straight line from the origin through $M$ to determine its intersection with the angle scale.
Example 18.16—Determination of the load impedance from the reflection coefficient.
The converse problem of determining the normalized load impedance for a given (say, experimentally determined) reflection coefficient at the load is equally simple. For example, according to Fig. 18.16, for $\rho = 0.8e^{-j30^\circ}$, that is, $|\rho| = 0.8$ and $\theta_\rho = -45^\circ$, we obtain $z \simeq 0.75 - j2.20$. So if, say, $Z_0 = 60 \Omega$, the load impedance is $Z_L = z \cdot Z_0 \simeq (45.0 - j132) \Omega$.

In addition to these two simple applications of the Smith chart, there are several more sophisticated ones. Perhaps the most important is that of determining the input impedance of a line of a given length and characteristic impedance, terminated by a given load impedance. This problem can be solved by means of Eq. (18.28), but an approximate solution using the Smith chart is quite simple. Let us consider the position along the line at a distance $\zeta$ from the load, as in Fig. 18.7. We first normalize $Z(\zeta)$ given in Eq. (18.25) with respect to $Z_0$:

$$z(\zeta) = \frac{Z(\zeta)}{Z_0} = \frac{1 + \rho(0)e^{-j2\beta\zeta}}{1 - \rho(0)e^{-j2\beta\zeta}}. \tag{18.48}$$

For $\zeta = 0$, $z(\zeta)$ is identical to $z = Z_L/Z_0$. But $z(\zeta)$ is of exactly the same form as $z$, except that $\rho(0)$ in $z$ is replaced by $\rho(\zeta) = \rho(0)e^{-j2\beta\zeta}$:

$$z(\zeta) = \frac{1 + \rho(\zeta)}{1 - \rho(\zeta)} \quad \rho(\zeta) = \rho(0)e^{-j2\beta\zeta}. \tag{18.49}$$

Because $|\rho(0)| \leq 1$ and $|e^{-2j\beta\zeta}| = 1$, $|\rho(\zeta)| \leq 1$. So the chart for $z(\zeta)$ and $\rho(\zeta)$ is exactly the same as for $z$ and $\rho = \rho(0)$. Now, if we know $z$ (for example, $Z_L$ and $Z_0$), we can locate the point on the Smith chart that determines $\rho$ directly. To obtain $z(\zeta)$, however, $\rho(\zeta) = \rho e^{-j2\beta\zeta}$ is needed rather than $\rho$. But multiplying a complex number by $e^{-j2\beta\zeta}$ implies changing its angle by $-2\beta\zeta$, leaving its magnitude constant, which means that we simply have to rotate $\rho$ (corresponding to $z$) by $2\beta\zeta$, in the clockwise (negative) direction. Thus we obtain $\rho(\zeta)$ and can read $z(\zeta)$ directly from the chart.

Noting that

$$2\beta\zeta = \frac{2\pi}{\lambda} \cdot \zeta = \frac{2\pi}{\lambda} \cdot 2\pi, \tag{18.50}$$

it follows that an angle of rotation $2\pi$ corresponds to $\zeta = \lambda/2$. This must be so because we know from Eq. (18.28) that $z(\zeta) = z(\zeta + \lambda/2)$. To facilitate the rotation of $\rho$ by the proper angle, an additional scale around the Smith chart is provided, with 0.5 (wavelengths) corresponding to one complete revolution around the unit circle $|\rho| = 1$. In the chart shown in Fig. 18.16 this wavelength scale is designated by “wavelengths toward generator.” For some applications the same scale in the opposite (counterclockwise) direction is also useful and is designated by “wavelengths toward load” in Fig. 18.16.

So we have the following additional properties of the Smith chart related to the reflection coefficient $\rho(\zeta)$:

- Moving around the chart in the clockwise direction corresponds to moving down the line from the load toward the generator (the phase increases).
• Moving around the chart in the counterclockwise direction corresponds to moving down the line from the generator toward the load (the phase decreases).

• One full circle around the chart corresponds to 180 degrees of phase (or half a wavelength). This is because the phase of the reflection coefficient changes as $e^{\beta z}$, so everything repeats every half wavelength down a line.

**Example 18.17—Input impedance of a line terminated in an arbitrary impedance.** As an example, let us consider a line of characteristic impedance $Z_0 = 60 \, \Omega$ and of length $\xi = 0.40 \lambda$ at the frequency used. Let us suppose that $Z_L = (90 - j60) \, \Omega$, and that we wish to determine the input impedance of the line thus terminated, using the Smith chart.

First, $z = Z_L/Z_0 = 1.5 - j1$, and we start with this value in the chart. This point has to be rotated in a clockwise direction by 0.4 units on the wavelength scale. Therefore we draw a straight line from the center of the chart through the point $z = 1.5 - j1$. The intersection of this line and the “wavelengths toward generator” scale is at 0.308 on the scale. We add 0.40 to this and get 0.708. This is 0.208 farther than point 0.000 on the scale. We draw a straight line from the 0.208 point of the wavelength scale toward the chart center and measure along this line the distance of the point $z = 1.5 - j1$ from the center. The point found in this way determines $z(\xi) = z(0.4\lambda)$. From the chart we find that $z(0.4\lambda) \simeq (1.83 + j0.95)$. So the input impedance of a 0.4\lambda long 60-\Omega line terminated with $Z_L = (90 - j60) \, \Omega$ is $Z(0.4\lambda) = Z_0 z(0.4\lambda) \simeq (110 + j57.0) \, \Omega$.

**Example 18.18—Examples of matching by transmission-line segments.** As already mentioned, at high frequencies (above about 100 MHz) it is not simple to make passive elements like resistors, capacitors, inductors, and transformers. For example, shunt (parallel) susceptance of interwinding capacitances of coils at these frequencies becomes pronounced and may completely distort the frequency behavior of the inductor. Shorted or open sections of transmission lines do not have this problem, so they are frequently used to replace reactive circuit elements in such cases. Such transmission-line reactive elements are often used for matching a high-frequency load to a desired impedance.

Another possible use of transmission-line segments for matching is as components for matching a load to a transmission line of a given characteristic impedance. Three principal ways of using transmission-line matching sections are sketched in Fig. 18.17.
Suppose the load impedance is \( Z_L = R_L + jX_L \) and the transmission-line characteristic impedance is \( Z_0 \neq R_L \). We can attempt to match the load to the line by following two steps:

1. Add a shorted line section in parallel to the load (a "stub" labeled 2 in Fig. 18.17), such that the admittance (and impedance) of the combination becomes real. Let the impedance of the combination be \( Z'_L \).

2. Add a quarter-wavelength matching line section like that labeled 1 in Fig. 18.17 to match the load \( Z'_L \) to \( Z_0 \).

In some instances, a matching line section of length different than quarter wavelength may do the entire job when it transforms the load impedance to approximately \( Z_0 \) without the stub at the load.

Finally, it is possible to add another stub, labeled 3 in Fig. 18.17, at a convenient location along the line to improve matching. Although all such problems can be solved with ease by programmable calculators or computers, they can also be solved simply using the Smith chart. Several specific examples of matching are given in the problems at the end of the chapter.

These examples illustrate only some of the simplest applications of the Smith chart. Several others will be found in the problems at the end of the chapter. Applications of the Smith chart are much more diverse than these examples suggest. The chart can also be used for analyzing plane waves perpendicularly incident on a plane boundary surface, for analyzing lossy lines, and for many other problems. Even though we can use a computer to perform such tasks, the Smith chart is useful for presenting the results and getting an intuitive feel for what the analysis tells us.

Questions and problems: Q18.22 and Q18.23, P18.31 to P18.40

18.7 Chapter Summary

1. A transmission line is an electromagnetic structure, so strictly speaking it should be analyzed by means of the field equations. However, a very short segment of a line can be approximated by a simple circuit, and the complete line by a chain of such circuits. Consequently, transmission lines can also be analyzed using circuit theory.

2. The voltage and current along transmission lines have a property not encountered in "normal" circuits: they move along the line with a certain velocity. These moving voltages and currents are known as voltage and current waves.

3. If the line is infinitely long, the ratio of the voltage and current waves propagating in one direction along the line, at any point and at any instant, is constant. This constant is known as the line characteristic impedance, and it depends only on how the line is made (dimensions and materials).

4. For lossless air lines, the velocity of propagation of voltage and current waves along them equals the velocity of light in a vacuum, whereas in all other cases this velocity is smaller.
5. The input impedance of open- or short-circuited transmission-line segments is purely reactive. Therefore such segments are used at high frequencies as capacitors and inductors.

6. Transmission-line segments act as specific, frequency-dependent transformers of impedances of loads connected at their end. This, combined with adding appropriate segments (stubs) of shorted (or open) lines in parallel, can be used for matching a transmission line to the load it is terminated in.

QUESTIONS

Q18.1. Why is it not practically possible to obtain a coaxial cable of characteristic impedance \( Z_0 = 500 \, \Omega \)? Can you have a two-wire line of this characteristic impedance?

Q18.2. Assume that a transmission line is made of two parallel, highly resistive wires. Can this line be analyzed using fundamental transmission-line equations? Explain.

Q18.3. A coaxial cable is filled with water. Does it represent a transmission line? Explain.

Q18.4. Two wires several wavelengths long serve as a connection between a generator and a receiver. The distance between the wires is small but not constant, varying as a smooth function of the coordinate along the line. Can you use the transmission-line equations for the analysis of this line? Explain.

Q18.5. Explain how you can obtain (1) a forward wave only; (2) a backward wave only along a transmission line.

Q18.6. Describe at least three ways of obtaining simultaneously a forward and a backward sinusoidal wave of the same amplitude along a transmission line.

Q18.7. Why can you replace an infinitely long end of a transmission line with a resistor of resistance equal to the line characteristic impedance?

Q18.8. Can a voltage (or a current) wave along a transmission line be described by the expression of the form \( u(x,y) \), where \( u(x,y) \) is a function of the product of the arguments \( x = (t-z/c) \) and \( y = (t+z/c) \)? Explain.

Q18.9. Can we adopt the negative instead of positive value of the square root in Eq. (18.7) for the velocity of wave propagation along transmission lines? Explain.

Q18.10. Why must the exponent of the forward voltage and current waves in Eqs. (18.13) and (18.17) be negative? Why must those of the backward waves be positive?

Q18.11. Is the wavelength along an air line greater or less than that in the same line filled with a dielectric? What is the answer if the dielectric has relative permeability greater than one? Explain.

Q18.12. What are the SI units for the following quantities: (1) the attenuation constant \( \alpha \), (2) the phase constant \( \beta \), (3) the reflection coefficient \( \rho \), and (4) the voltage standing-wave ratio (VSWR)?

Q18.13. What is the magnitude of the reflection coefficient, \( |\rho| \), and of the VSWR, for which one half of the power of the incident wave is transferred to the load?

Q18.14. Why is the voltage at the termination \( Z \) of a transmission line with characteristic impedance \( Z_0 \) equal to \( V = 2V_+Z/(Z+Z_0) \)?

Q18.15. What are the input impedances to lossless lines of lengths \( \lambda/4 \) and \( \lambda/2 \), if they are (1) open-circuited or (2) short-circuited?
Q18.16. Can a resistive load of any resistance $R$ be matched in practice to a transmission line of characteristic impedance $Z_0$? Explain.

Q18.17. The characteristic impedance of a lossy line in Eq. (18.32) is real if $R' = 0$ and $G' = 0$. Can it be real for some other relation between $R'$, $L'$, $G'$, and $C'$? Explain.

Q18.18. Why could we not use simple transmission-line analysis when calculating the step response of an inductor, as in Fig. Q18.18?

![Figure Q18.18 Calculating the step response of an inductor](image)

Q18.19. If you had a break in the dielectric of a cable causing a large shunt conductance, what do you expect to see reflected if you excite the cable with a short pulse (practical delta function)?

Q18.20. If you had a break in the outer conductor of a cable, causing a large series resistance, what do you expect to see reflected if you excite the cable with a short pulse (imperfect delta function)?

Q18.21. What do the reflected waves off a series inductor and shunt capacitor in the middle of a transmission line look like for a short pulse excitation, assuming that $\omega L \gg Z_0$ and $\omega C \gg 1/Z_0$?

Q18.22. Using the Smith chart, determine the complex reflection coefficient on a 60-\(\Omega\) line if it is terminated by (1) 80 \(\Omega\), (2) (30 – j40) \(\Omega\), or (3) (40 + j90) \(\Omega\).

Q18.23. Using the Smith chart, determine the terminating impedance of a 70-\(\Omega\) line if it was found experimentally that the complex voltage reflection coefficient is (1) 0.8, (2) $0.2e^{-j\pi/4}$, or (3) $0.5e^{j\pi/3}$.

PROBLEMS

P18.1. Given a high-frequency RG-55/U coaxial cable with $a = 0.5$ mm, $b = 2.95$ mm, $\varepsilon_r = 2.25$ (polyethylene), and $\mu_r = 1$, find the values for the capacitance and inductance per unit length of the cable.

P18.2. Assume that the coaxial cable from problem P18.1 is not lossless but that the losses are small, resulting in an attenuation constant in decibels per meter at 10 GHz of $\alpha = 0.5$ dB/m. Assuming the dielectric in the cable to be perfect, find the resistance per unit length that causes the losses in the conductors.

P18.3. The distance $d$ between wires of a lossless two-wire line is a smooth, slowly varying function of the coordinate $z$ along the line so that the line capacitance and inductance per unit length, $L'$ and $C'$, are also smooth functions of $z$. $L' = L'(z)$, and $C' = C'(z)$. Derive the transmission-line equations for such a nonuniform transmission line. Check
if these equations become the transmission-line equations (18.4) for \( L'(z) \) and \( C'(z) \) constant.

**P18.4.** Using circuit theory, analyze approximately a matched, lossless, air-filled coaxial transmission line of length \( l = \lambda \) and conductor radii \( a = 1 \text{ mm} \) and \( b = 3 \text{ mm} \) as a connection of \( n \) cells of the type in Fig. 18.3b, for \( n = 1, 2, \ldots , 20 \). Such a circuit-theory approximation to transmission lines is known as an artificial transmission line. Note that an artificial transmission line can be analyzed as a simple ladder network. Assume the artificial line to be terminated in the actual characteristic impedance, and compare current in series-concentrated inductive elements and voltage across parallel concentrated capacitive elements with exact results. Solve the problem so that you can vary \( L', C' \), and \( n \).

**P18.5.** Noting that \( c = \frac{1}{\sqrt{\varepsilon \mu}} \) for all transmission lines in Table 18.1, prove that for these lines the inductance per unit length and the characteristic impedance of a lossless transmission line can be expressed in terms of \( c \) and \( C' \).

**P18.6.** Express \( V(z) \) in Eq. (18.13) and \( I(z) \) in Eq. (18.17) for lossless lines in terms of the sending-end voltage \( V(0) \) and sending-end current \( I(0) \).

**P18.7.** Prove that it is possible to obtain the characteristic impedance of any lossless line by measuring the input impedance of a section of the line when it is open-circuited, and when it is short-circuited.

**P18.8.** A lossless line of characteristic impedance \( Z_{01} \) and length \( l_1 \) is terminated in an impedance \( Z_L \). The line serves as a load for another lossless line of characteristic impedance \( Z_{02} \) and length \( l_2 \). The dielectric in both lines is air and the angular frequency of the current is \( \omega \). Determine general expressions for the input impedance of the second line, the reflection coefficient in both lines, and the voltage standing-wave ratio in both lines.

**P18.9.** A short and then an open load are connected to a 50-\( \Omega \) transmission line at \( z = 0 \). Make a plot of the impedance, normalized voltage ("normalized" means that you divide the voltage by its maximal value to get a maximum normalized voltage of 1), and normalized current along the line up to \( z = -3\lambda/2 \) for the two cases.

**P18.10.** A lumped capacitor is inserted into a transmission-line section, as shown in Fig. P18.10. Find the reflection coefficient for a wave incident from the left. Assume the line is terminated to the right so that there is no reflection off the end of the line. Find a simplified expression that applies when \( C \) is small. The characteristic impedance of the line is \( Z_0 \).

![Figure P18.10](image)

![Figure P18.11](image)

**P18.11.** Repeat problem P18.10 assuming that a lumped resistor is inserted into a transmission-line section as shown in Fig. P18.11. Find a simplified expression that applies when \( R \) is small.
P18.12. Repeat problem P18.10 assuming that a lumped inductor is inserted into a transmission-line section as shown in Fig. P18.12. Find a simplified expression that applies when $L$ is small.

\[ \begin{align*}
& \text{Figure P18.12 A series coil in a line} \\
& \text{Figure P18.13 A shunt resistor in a line}
\end{align*} \]

P18.13. Repeat problem P18.10 assuming that a lumped resistor is inserted into a transmission-line section as shown in Fig. P18.13. Find a simplified expression that applies when $R$ is large.

P18.14. A 50-Ω transmission line needs to be connected to a 100-Ω load. The setup is used at 1 GHz. What would you connect between the line and the load to have no reflected voltage on the line? *Lossless Elements*

P18.15. In problem P18.14, the load is a 100-Ω resistor but the leads are long and represent a 2 nH inductor in series with the resistor. How would you get rid of the reflected voltage on the line in this case?

P18.16. Find the transmission coefficient for the transmission line in Fig. P18.10.

P18.17. Find the reflection and transmission coefficients for the transmission line in Fig. P18.17. Because the reflection coefficient is defined by voltage, the power is given by its square. What are the reflected and transmitted power equal to? Does the power balance make sense?

\[ \begin{align*}
& \text{Figure P18.17 Two resistors in a line}
\end{align*} \]

P18.18. Derive the normalized input impedance (i.e., the impedance divided by $Z_0$) for a section of line that is $n\lambda/8$ long and shorted at the other end, for $n = 1, 2, 3, 4,$ and $5$. Plot the impedance as a function of electrical line length from the load (length measured in wavelengths along the line).

P18.19. Repeat problem P18.18 for an open-ended line.

P18.20. Find the total current and voltage at the beginning of a $\lambda/4$ shorted transmission line of characteristic impedance $Z_0$. What circuit element does this line look like? Plot the
total current and voltage as a function of electrical line length from the load (length measured in wavelengths along the line).

**P18.21.** Repeat the previous problem for an open-ended line.

**P18.22.** Find the reflection and transmission coefficients for an ideal $n : 1$ transformer, as in Fig. P18.22, where $n$ is the voltage transformation ratio.

![Figure P18.22 An ideal transformer](image)

**P18.23.** Find the input impedance for the circuit in Fig. P18.23.

![Figure P18.23 Impedance of a line with a shunt stub](image)

**P18.24.** A coaxial transmission line with a characteristic impedance of 150 $\Omega$ is 2 cm long and is terminated in a load impedance of $Z = 75 + j150$ $\Omega$. The dielectric in the line has a relative permittivity of $\varepsilon_r = 2.56$. Find the input impedance and VSWR on the line at $f = 3$ GHz.

**P18.25.** Match a 25-$\Omega$ load to a 50-$\Omega$ line using (1) a single quarter-wave section of line, or (2) two quarter-wave line sections.

**P18.26.** Match a purely capacitive load, $C = 10$ pF, to a 50-$\Omega$ line at 1 GHz. How many different ways can you think of doing this?

**P18.27.** Calculate and plot magnitude and phase of $\rho(f)$ between 1 and 3 GHz for a 50-$\Omega$ open transmission line that is $\lambda/4$ long at 2 GHz.

**P18.28.** If you had a cable like the one in problem P18.2 spanning the Atlantic and you sent a continuous signal of 1-MW power from the United States to England, how much power approximately would you get in England? (Look up the approximate distance across the Atlantic in an atlas if you need to.)

**P18.29.** A printed-circuit board trace in a digital circuit is excited by a voltage $v(t)$, as in Fig. P18.29. Derive an equation for the coupled (cross-talk) signal on an adjacent line, $v_c(t)$, assuming the adjacent line is connected to a load at one end and a scope (infinite
impedance) at the other end so that no current flows through it. (*Hint:* the coupling is capacitive and you can approximate it by a capacitor between the two traces and use circuit theory.)

![Figure P18.29 An example of two coupled lines](image)

![Figure P18.30 Measuring the reflected wave from a complex load](image)

**P18.30.** Derive the expression \( t_1 = 0.69 t_L \) discussed in Example 18.14. This expression shows a practical way to measure the time constant of the reflected wave for the case of complex loads, as in Fig. P18.30.

**P18.31.** Trace the procedure for solving problem P18.8 by means of the Smith chart.

**P18.32.** The reciprocals of complex numbers can be determined easily from the Smith chart. Starting with Eq. (18.49), deduce how this can be done.

**P18.33.** A fixed, known complex impedance \( Z_L \) is to be connected to a lossless line having a characteristic impedance \( Z_0 \). Show that it is possible to eliminate the reflected wave along the line if an appropriate length of the same line, assumed to be short-circuited, is connected at an appropriate place on the line near \( Z_L \) (see Fig. P18.33).

![Figure P18.33 A configuration for matching a load to a transmission line](image)
P18.34. What circuit element corresponds to the point on the Smith chart that is defined by the intersection of the circle \( r = 1 \) and the arc \( jx = j1.2 \) at 1 GHz, if the normalizing impedance is 50 \( \Omega \)?

P18.35. What circuit element corresponds to the point on the Smith chart that is defined by the intersection of the circle \( r = 1 \) and the arc \( jx = -j0.4 \) at 500 MHz, if the normalizing impedance is 50 \( \Omega \)?

P18.36. At the load of a terminated transmission line of characteristic impedance \( Z_0 = 100 \, \Omega \), the reflection coefficient is \( \rho = 0.56 + j0.215 \). What is the load impedance?

P18.37. A 50-\( \Omega \) line is terminated in a load impedance of \( Z = 80 - j40 \, \Omega \). Find the reflection coefficient of the load and the VSWR.

P18.38. A 50-\( \Omega \) slotted line measurement (see Example 18.10) was done by first placing a short at the place of the unknown load. This results in a large VSWR on the line with sharply defined voltage minima. On an arbitrarily positioned scale along the air-filled coaxial line, the voltage minima are observed at \( z_c = 0.1, 1.1, \) and \( 2.1 \) cm. The short is then replaced by the unknown load, the VSWR is measured to be 2, and the voltage minima (not as sharp as with the short termination) are found at \( z = 0.61, 1.61, \) and \( 2.61 \) cm. Use the Smith chart to find the complex impedance of the load. Explain all your steps.

P18.39. Use a shorted parallel stub to match a 200-\( \Omega \) load to a 50-\( \Omega \) transmission line. Include a Smith chart plot with step-by-step explanations.

P18.40. A load consists of a 100-\( \Omega \) resistor in series with a 10-nH inductor at 1 GHz. Use an open single stub to match it to a 50-\( \Omega \) line.