19

Maxwell’s Equations

19.1 Introduction

This chapter is devoted to the extension of the equations we have derived so far to the most general equations for the electromagnetic field. These general equations are known as Maxwell’s equations. Any engineering problem that includes electromagnetic fields is solved starting from these equations, although in some instances the application may not be obvious. For example, ac circuits are in fact described by an approximation of Maxwell’s equations valid for specific fields existing in such circuits.

We will see that Maxwell’s equations can be written in two forms: integral and differential. We will also see that numerous general conclusions follow from these equations. For example, the problem of energy transfer by means of an electromagnetic field can be understood and solved only if we start from Maxwell’s equations and derive what is known as Poynting’s theorem. General boundary conditions will also be derived. Finally, we will show that in many important instances electromagnetic field vectors can be derived from auxiliary functions, known as potentials.

This is probably the most important chapter in the entire book. It unifies all the concepts we have studied so far. It also adds the concept of displacement current that couples Gauss’, Ampère’s, and Faraday’s laws with the current continuity and conservation of magnetic flux equations. Maxwell’s equations enable us to solve many practical engineering problems that deal with electromagnetic fields.
19.2 Displacement Current

We know from Faraday's law that a time-varying magnetic field is always accompanied by a time-varying electric (induced) field. This also means that a time-varying electric field is accompanied by a time-varying magnetic field. We have learned so far that sources of a magnetic field are electric currents. From the preceding inverse statement, we can say that a *time-varying magnetic field* is not caused solely by time-varying electric currents but also by a *time-varying electric field*.

This conclusion is the essence of Maxwell's contribution to the theory of electricity and magnetism. To stress that this time-varying electric field is the source of a magnetic field, as is a current, a quantity tightly connected with time variation of the electric field is termed the *displacement current*, even though it is not a current in the usual sense.

Consider a circuit containing an air-filled parallel-plate capacitor and with time-varying current flowing through it, as sketched in Fig. 19.1. Imagine two surfaces, $S_1$ and $S_2$, shown in the figure. The surface $S_1$ intersects a part of the wire. The surface $S_2$ intersects one electrode of the capacitor only.

If we apply the current continuity equation [Eq. (10.14)],

$$\int_S \mathbf{j} \cdot d\mathbf{S} = -\frac{d}{dt} \int_S \rho \, d\mathbf{v},$$  \hspace{1cm} (10.14)

to $S_1$, we find that it is satisfied because a current $i(t)$ enters the surface, the same current leaves the surface, and there is no charge accumulation along the enclosed wire segment. If, however, we apply Eq. (10.14) to $S_2$, we are working with a current entering $S_2$, but no current leaving this surface. Instead, we have an increase of charge in $S_2$ such that Eq. (10.14) is satisfied.

Suppose we wish to express the general equation for current continuity in Eq. (10.14) as a surface integral on the left-hand side, and a zero on the right-hand side. This can be done easily if we recall Gauss' law in Eq. (7.20). The volume integral

![Figure 19.1 Circuit containing an air-filled capacitor and with time-varying current flowing through it](image-url)
in Eq. (10.14) is precisely \( Q_{\text{free}} \) in Eq. (7.20), except that this charge now varies in time. So instead of Eq. (10.14) we can write an equivalent equation

\[
\oint_S \mathbf{J} \cdot \, d\mathbf{S} = -\frac{d}{dt} \oint_S \mathbf{D} \cdot d\mathbf{S}.
\]  

(19.1)

If we assume that the surface \( S \) is not varying in time, the time derivative can be introduced under the integral sign to act on the vector \( \mathbf{D} \) only. Noting that the surface integrals on the two sides of the equation refer to the same surface, we can write Eq. (19.1) in the form

\[
\oint_S \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S} = 0.
\]  

(19.2)

We have arrived at an interesting conclusion: the flux through a closed surface of the vector sum \( (\mathbf{J} + \partial \mathbf{D}/\partial t) \) is always zero. The expression \( \partial \mathbf{D}/\partial t \) has the dimension of current density. It is therefore termed the displacement current density.

We know that if the flux of a vector function through any closed surface is zero, then the flux of that vector through all open surfaces bounded by the same contour is the same. Consider the contour \( C \) indicated in Fig. 19.1, and two surfaces bounded by the contour, \( S_1 \) and \( S_2 \). The surface \( S_1 \) cuts the wire, so the flux of \( (\mathbf{J} + \partial \mathbf{D}/\partial t) \) through it is simply \( i(t) \). The surface \( S_2 \) passes between the capacitor electrodes and does not cut the wire. Therefore, there is no current through that surface, and the flux of \( (\mathbf{J} + \partial \mathbf{D}/\partial t) \) equals that of vector \( \partial \mathbf{D}/\partial t \) through it. We will now show that these two integrals are equal.

Open surfaces \( S_1 \) and \( S_2 \) make the closed surface \( S \). The flux of \( (\mathbf{J} + \partial \mathbf{D}/\partial t) \) through \( S \) is calculated with respect to the outward unit vector normal to \( S \). Recall the right-hand rule of defining a unit vector normal to a surface defined by a contour. The flux through the part \( S_1 \) of \( S \) is calculated with respect to the outward normal, but the flux through the part \( S_2 \) of \( S \) should be calculated with respect to the opposite normal. Consequently Eq. (19.2) yields

\[
\int_{S_1} \mathbf{J} \cdot d\mathbf{S} = \int_{S_2} \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}.
\]

This could be interpreted as if the conductive current in the metallic wire continues between the capacitor plates in the form of the displacement current. In other words, if we consider a time-varying conductive current only, it has sources and sinks. The total current (the sum of conductive and displacement currents), however, does not have sources and sinks, but rather closes onto itself, as a dc current. With this in mind, Maxwell postulated that in time-varying fields the source of the magnetic field is not solely the conductive current, but rather the total current, the density of which is \( (\mathbf{J} + \partial \mathbf{D}/\partial t) \).

From Ampère's law we know that the line integral of the magnetic field intensity vector, \( \mathbf{H} \), along a closed contour equals the current through any surface defined by the contour. From the reasoning we just did, we see that it is also equal to the flux of the displacement current through the contour (i.e., through a surface bounded by
the contour):

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_{S_1} \mathbf{J} \cdot d\mathbf{S} = \int_{S_2} \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}. \quad (19.3)$$

This equation tells us that if we wish Ampère's law to be valid for time-varying currents, we must replace \( \mathbf{J} \) in it by \( (\mathbf{J} + \partial \mathbf{D}/\partial t) \). So the generalized Ampère's law has the form

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S} \quad \text{(generalized Ampère's law).} \quad (19.4)$$

This is the fundamental contribution of Maxwell, which can be interpreted as follows: the displacement current produces a magnetic field according to the same law as "normal" current. We will see that the addition of the displacement current density in Ampère's law has far-reaching consequences. For example, without it we cannot explain the existence of electromagnetic waves. An electromagnetic wave is a moving electromagnetic field that, once created by charges and currents, continues to exist with no connection whatsoever to the charges and currents that created it.

Example 19.1—Displacement current density in dielectrics and in a vacuum. Since \( \mathbf{D} = (\varepsilon_0 \mathbf{E} + \mathbf{P}) \), the displacement current density, \( \partial \mathbf{D}/\partial t \), can be written in the form

$$\frac{\partial \mathbf{D}}{\partial t} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t}. \quad (19.5)$$

We know that the polarization vector, \( \mathbf{P} \), represents the transfer of real charge per unit area normal to vector \( \mathbf{P} \). It is measured in C/m². Therefore the expression \( \partial \mathbf{P}/\partial t \) is in A/m², and represents a real current density, resulting from the motion of the polarization charges. This part of the displacement current density is termed the displacement current density in the dielectric, or frequently, the density of polarization current.

The other part of the displacement current density, \( \varepsilon_0 \partial \mathbf{E}/\partial t \), is measured in the same units, A/m², but it does not represent any motion of real charges. This is the displacement current density in a vacuum. This part of the displacement current can be very misleading, however, if one does not keep in mind its physical meaning: the time-varying electric field is the source of the time-varying magnetic field. In other words, as far as the source of the magnetic field is concerned, \( \varepsilon_0 \partial \mathbf{E}/\partial t \) is completely equivalent to an electric current of the same density, although it does not represent any real motion of electric charges.

Questions and problems: Q19.1, P19.1 to P19.3

19.3 Maxwell's Equations in Integral Form

We are now ready to formulate the general equations of the electromagnetic field in integral form. In fact, what we need to do is to review all the equations we have postulated or derived, and make sure they are not contradictory. If they do not contradict each other, we can, following Maxwell, postulate them to be true for all electromagnetic fields. The sole criterion for the validity of these equations is, of course, experiment. Ever since Maxwell postulated in the 1860s the equations that bear his name,
no experimental evidence has indicated even the slightest disagreement with these equations.

We now write the integral form of the four most general equations we have derived. With no particular reason except that it is customary, we start with Faraday's law in Eq. (14.6) for a fixed contour, so that the time derivative can be introduced under the integral sign. This equation is usually followed by the generalized Ampère's law. Gauss' law in Eq. (7.20), in which the total free charge enclosed by a closed surface is replaced by a volume integral of the charge density, is the third equation. The last equation is the law of conservation of magnetic flux.

Thus Maxwell's equations in integral form are

\[ \oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}, \]  \hspace{1cm} (19.5)

*Faraday's law for a fixed contour, Eq. (14.6) = Maxwell's first equation*

\[ \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}, \]  \hspace{1cm} (19.6)

*Generalized Ampère's law, Eq. (19.4) = Maxwell's second equation*

\[ \oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho \, dv, \]  \hspace{1cm} (19.7)

*Gauss' law, Eq. (7.20) = Maxwell's third equation*

\[ \oint_S \mathbf{B} \cdot d\mathbf{S} = 0. \]  \hspace{1cm} (19.8)

*Law of conservation of magnetic flux, Eq. (12.11) = Maxwell's fourth equation*

Finally, we add to these equations the current continuity equation,

\[ \oint_S \mathbf{J} \cdot d\mathbf{S} = -\int_v \frac{\partial \rho}{\partial t} \, dv. \]  \hspace{1cm} (19.9)

*Current continuity equation, Eq. (10.15) = law of conservation of electric charge*
These equations can be paraphrased as follows. Equation (19.5) tells us that a time-varying magnetic field is a source of an (evidently time-varying) electric field. Equation (19.6) states that the sources of a magnetic field are electric currents and a time-varying electric field. According to Eq. (19.7), the only source that produces a nonzero flux through a closed surface of the electric displacement vector are free electric charges. Finally, Eq. (19.8) can be interpreted as stating that no analogue of free electric charges exists for a magnetic field. Equation (19.9) is not a field equation, but the law of conservation of electric charge must be satisfied by all real sources of the electromagnetic field.

If we try to find any logical deficiencies in these equations, we will see that there are none, in spite of the fact that they have been derived separately, for specific types of fields. For this reason we postulate that these equations are always valid and represent the equations of the general electromagnetic field.

There are numerous applications of Maxwell’s equations in integral form. One group of applications relates to the derivation of some general conclusions about electromagnetic fields. One of the most important applications of this type is the derivation of general boundary conditions.

Example 19.2—General boundary conditions. We know that boundary conditions are relations between values of any field quantity at two close points on the two sides of a surface between two different media. For the four basic field vectors, \( \mathbf{E}, \mathbf{H}, \mathbf{D}, \) and \( \mathbf{B} \), they are but special forms of the integral Maxwell’s equations (19.5) to (19.8).

In order to be able to derive them in the most usual form, we need to consider also the possibility of surface currents. We shall see in the next chapter that at high frequencies, currents in good conductors are distributed essentially over conductor surfaces, and are practically surface currents. This is why we need to include surface currents in boundary conditions, and to specialize the boundary conditions at the surface of a “perfect” conductor.

If one of the two media on two sides of a boundary surface is a perfect conductor, let it be medium 2. Inside a perfect conductor there can be no electric field (it would result in infinite current density). We know that a time-varying electric field is accompanied by a time-varying magnetic field. Therefore, inside a perfect conductor, there can also be no time-varying magnetic field.

We derived boundary conditions in the electrostatic field starting, in fact, from Eq. (19.5) (with zero right-hand side), and from Eq. (19.7). Does the nonzero right-hand side in Eq. (19.5) change anything? Recall that in the derivation of the boundary condition for the tangential components of vector \( \mathbf{E} \) we assumed that the contour was infinitely narrow. Therefore, the flux of vector \( \partial \mathbf{B} / \partial t \) is zero also if we start from Eq. (19.5). On the other hand, Eq. (19.7) is the same as Gauss’ law in electrostatics. So we conclude that in any electromagnetic field, on the two sides of any boundary surface, both electrostatic conditions, Eqs. (7.26) and (7.27), remain valid. If one of the media is a perfect conductor, these equations take the forms that are also valid in electrostatics (but for any, not necessarily perfect, conductor):

\[
\mathbf{E}_1 \text{tang} = \mathbf{E}_2 \text{tang}, \quad \text{or} \quad \mathbf{E}_\text{tang} = 0 \quad \text{on surface of perfect conductor,} \quad (19.10)
\]

*General boundary condition for tangential components of \( \mathbf{E} \)*
\[ D_{1\text{norm}} - D_{2\text{norm}} = \sigma, \quad \text{or} \quad D_{\text{norm}} = \sigma \quad \text{on surface of perfect conductor.} \quad (19.11) \]

(General boundary condition for normal components of \( D \))

The condition for the tangential components of the magnetic field intensity vector was derived from Ampère's law, applied to an infinitely narrow contour. Displacement current through such a contour is zero, but conduction current may be nonzero if there is a surface current on the boundary.

Consider Fig. 19.2 and assume the surface-current density vector \( J_s \) to be locally in the \( y \) direction. The magnetic field of these currents is then in the \( x \) direction, as indicated. The current through the narrow rectangular contour in the figure, which is in the \( x-z \) plane, i.e., normal to \( J_s \), is \( J_s \Delta l \). The integral of vector \( H \) around the contour is \( (H_{1x} - H_{2x})\Delta l \). Noting that the unit vector normal to the boundary is directed into medium 1, from the integral form of Ampère's law we obtain

\[ H_{1\text{tang}} - H_{2\text{tang}} = J_s \times n, \quad \text{or} \quad H_{\text{tang}} = J_s \times n \quad \text{on surface of perfect conductor.} \quad (19.12) \]

(General boundary condition for tangential components of \( H \)—see Fig. 19.2)

The condition for the normal components of vector \( B \), Eq. (13.8), also remains the same, since it was derived from the law of conservation of magnetic flux, Eq. (19.8). If medium 2 is a

![Figure 19.2 Surface current on boundary between two media. The magnetic field due to this current is locally normal to the surface-current density vector.](image)
perfect conductor, no field is there, and we have

\[ B_{1\text{norm}} = B_{2\text{norm}}, \quad \text{or} \quad B_{\text{norm}} = 0 \quad \text{on surface of perfect conductor.} \quad (19.13) \]

(General boundary condition for normal components of \( \mathbf{B} \))

It is worthwhile repeating what we need boundary conditions for. These equations are, in fact, Maxwell’s equations specialized to boundary surfaces. Therefore in a medium consisting of several bodies of different properties, the field transition from one body to the adjacent body, through a boundary surface, must be as required by the boundary conditions. If this were not so, such an electromagnetic field could not be a real field, because it would not satisfy the field equations everywhere.

Questions and problems: Q19.2 to Q19.4

19.4 Maxwell’s Equations in Differential Form

Maxwell’s equations in integral form, Eqs. (19.5) to (19.8), can be transformed into a set of differential equations, known as Maxwell’s equations in differential form. They can easily be obtained from the integral forms by applying the Stokes’s and the divergence theorems of vector analysis. (If necessary, consult Appendix 1, Sections A1.4.6 and A1.4.7, to refresh your knowledge of these two theorems before proceeding further.)

Consider the first and second Maxwell’s equations. By Stokes’s theorem, the line integral of \( \mathbf{E} \) in the first equation can be transformed into the flux of the vector curl \( \mathbf{E} = \nabla \times \mathbf{E} \) through any surface bounded by the contour \( C \). Therefore, instead of Eq. (19.5) we can write the equivalent equation

\[ \int_S \nabla \times \mathbf{E} \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}. \quad (19.14) \]

The two surfaces have the same boundary contour, but they may or may not be the same. If they are the same, any surface bounded by \( C \) can be chosen. Such an equation can be satisfied, however, only if the integrands in the two integrals are equal at all points, that is, if \( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \). Maxwell’s second equation can be transformed in exactly the same manner.

The third and fourth Maxwell’s equations can also be written in an equivalent form from which we can obtain their differential counterparts. For example, apply the divergence theorem to the left side of Eq. (19.7), to obtain

\[ \int_V \nabla \cdot \mathbf{D} \, dv = \int_V \rho \, dv. \quad (19.15) \]

Note that the domains \( v \) on the two sides of the equation are the same. This equation can be satisfied for any domain \( v \) only if the integrands are equal at all points, that is, if \( \nabla \cdot \mathbf{D} = \rho \). In the same manner, we can transform the fourth Maxwell’s equation.
From these derivations, Maxwell's equations in differential form read

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$  \hspace{1cm} (19.16)

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$  \hspace{1cm} (19.17)

$$\nabla \cdot \mathbf{D} = \rho$$  \hspace{1cm} (19.18)

$$\nabla \cdot \mathbf{B} = 0.$$  \hspace{1cm} (19.19)

(Maxwell's equations in differential form)

Let us add here the current continuity equation in differential form, obtained in the same manner as the last two equations:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.$$  \hspace{1cm} (19.20)

(Current continuity equation in differential form)

To these equations (as well as to their integral counterparts) it is necessary to add the relationships between vectors (1) $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{P}$; (2) $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{M}$; and (3) $\mathbf{J}$ and $\mathbf{E}$:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \hspace{1cm} \mathbf{P} = \mathbf{P(}E\mathbf{)}$$  \hspace{1cm} (19.21)

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) \hspace{1cm} \mathbf{M} = \mathbf{M(}B\mathbf{)}$$  \hspace{1cm} (19.22)

$$\mathbf{J} = \mathbf{J(}E\mathbf{)}.$$  \hspace{1cm} (19.23)

For linear media, which are practically the only media we consider in this text, we have

$$\mathbf{D} = \epsilon \mathbf{E} \hspace{1cm} \mathbf{B} = \mu \mathbf{H} \hspace{1cm} \mathbf{J} = \sigma \mathbf{E}.$$  \hspace{1cm} (19.24)

(Constitutive relations for linear media)

Equations (19.21) to (19.23) and (19.24) are often referred to as the constitutive relations.

The differential Maxwell's equations are used for solving many electromagnetic problems. There are modern, extremely powerful numerical methods for solving these equations directly. As computers evolve, increasingly complex electromagnetic problems can be solved numerically in a reasonable amount of time.

It is interesting that the fourth equation follows from the first. Indeed, if we take the divergence of the left-hand and right-hand sides of Eq. (19.16), the left-hand side...
is equal to zero, because $\nabla \cdot (\nabla \times F)$ (divergence of the curl) of any vector functions $F$ is identically zero. Therefore, $\partial (\nabla \cdot \mathbf{B})/\partial t = 0$, which means that $\mathbf{B}$ does not depend on time. So if at any time in the past $\mathbf{B} = 0$ (and therefore also $\nabla \cdot \mathbf{B}$), which certainly was the case, then $\nabla \cdot \mathbf{B} = 0$ generally. In a similar manner one can prove that with the aid of the current continuity equation, the third Maxwell's equation follows from the second.

Questions and problems: Q19.5 to Q19.20

19.5 Maxwell's Equations in Complex (Phasor) Form

Maxwell's equations in differential form, Eqs. (19.16) to (19.19), are partial differential equations with three space coordinates and time as independent variables. Very often, the time variation of the sources is sinusoidal. If the medium is also linear, we know that all quantities vary in time sinusoidally. It is then possible to eliminate time from the equations, and thus simplify them. The procedure is very similar to that in circuit theory. The difference is that here we have vector quantities in addition to scalar quantities, and that these quantities are functions of space coordinates.

Quantities varying sinusoidally in time are often called time-harmonic. Their time dependence can be written in the form $\cos(\omega t + \varphi)$, where $\omega = 2\pi f$ is the angular frequency (in radians per second), $f$ is the frequency (in Hz), and $\varphi$ is the initial phase. In general, $\varphi$ is a function of coordinates. In the case of vector quantities, the initial phases of the three vector components at a point can be different.

Example 19.3—Complex field quantities. To understand the logic of complex representation of time-harmonic vectors, consider the $x$ component of a time-harmonic electric field of angular frequency $\omega$:

\[ E_x(x, y, z, t) = E_{x\text{ max}}(x, y, z) \cos(\omega t + \varphi(x, y, z)). \]  \hfill (19.25)

Euler's identity allows us to express the cosine as a sum of complex exponentials:

\[ \cos(\omega t + \varphi) = \frac{e^{j\omega t}e^{j\varphi} + e^{-j\omega t}e^{-j\varphi}}{2}, \]  \hfill (19.26)

where $j = \sqrt{-1}$ is the imaginary unit.

The time derivative of $E_x(x, y, z, t)$ can be written as

\[ \frac{\partial}{\partial t} E_x(x, y, z, t) = E_{x\text{ max}}(x, y, z) \frac{1}{2} \left( j\omega e^{j\omega t}e^{j\varphi(x,y,z)} - j\omega e^{-j\omega t}e^{-j\varphi(x,y,z)} \right). \]  \hfill (19.27)

All the quantities from Maxwell's equations can be expressed in this form. The equations written in such a way will contain some parts with a factor $e^{j\omega t}$, and the same parts with a factor $e^{-j\omega t}$. Because the two functions, $e^{j\omega t}$ and $e^{-j\omega t}$, are independent, the factors they multiply must be zero in order that the equations be satisfied at any $t$. In other words, instead of each equation, we get two equivalent complex equations. In these equations, time does not appear explicitly, and the time derivatives are replaced by $j\omega$, or $-j\omega$. 
Formally, one of these complex equations can be obtained from the initial equation by replacing all the cosines with $e^{j\omega t}$, and the other by replacing the cosines with $e^{-j\omega t}$. Then, after differentiating with respect to time, all factors with $e^{j\omega t}$ and $e^{-j\omega t}$ cancel out. Although both $e^{j\omega t}$ and $e^{-j\omega t}$ can be used, it is customary in electrical engineering to replace the cosine with $e^{j\omega t}$, so that the first time derivative is replaced by the factor $j\omega$, the second time derivative by the factor $-\omega^2$, etc.

A phasor quantity in electrical engineering is written as a complex root-mean square (rms) value. To stress that a quantity is a phasor or complex, the International Electronics Commission (IEC) recommends that it be underlined, as follows:

$$ A = A(x, y, z) e^{j\varphi(x, y, z)} = \frac{A_{\text{max}}(x, y, z)}{\sqrt{2}} e^{j\varphi(x, y, z)}. \quad (19.28) $$

The magnitude of the complex quantity is represented with the rms value instead of the maximum value because the expressions for average power and energy are conveniently expressed with rms values. Most instruments show rms values.

When dealing with complex vectors, it is important to keep in mind the following. A real vector has three components and, at any given moment, can be drawn as an arrow in space. The arrow describes the direction and magnitude of the vector. A complex vector is a set of six numbers, three real and three imaginary parts of its components. This is why, in general, a complex vector cannot be represented with an arrow.

After all these explanations, we can finally write down the simplest and most often quoted (but least general) form of Maxwell's equations—their complex form:

$$ \nabla \times \mathbf{E} = -j\omega \mathbf{B}. \quad (19.29) $$
$$ \nabla \times \mathbf{H} = \mathbf{J} + j\omega \mathbf{D}. \quad (19.30) $$
$$ \nabla \cdot \mathbf{D} = \rho. \quad (19.31) $$
$$ \nabla \cdot \mathbf{B} = 0. \quad (19.32) $$

(Maxwell's equations in complex form)

It is important to keep in mind that these equations are valid only for linear media. Otherwise, as explained, all quantities cannot simultaneously be time-harmonic.

In addition, we have the current continuity equation in complex form,

$$ \nabla \cdot \mathbf{J} = -j\omega \rho. \quad (19.33) $$

(Current continuity equation in complex form)

as well as the constitutive relations with complex vectors (phasors), and with complex permittivity, permeability, and conductivity,
\[ \mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}, \quad \mathbf{J} = \sigma \mathbf{E}. \quad (19.34) \]

\textit{(Constitutive relations in complex form)}

Questions and problems: Q19.21 to Q19.24, P19.4

19.6 Poynting’s Theorem

Poynting’s theorem is the mathematical expression of the law of conservation of energy as applied to electromagnetic fields.

To obtain an energy expression from Maxwell’s equations, we have to combine them in an appropriate way. We know that the expression \( \mathbf{J} \cdot \mathbf{E} \) is dissipated (Joule’s) power per unit volume. Note that \( \mathbf{E} \) stands for the electric field due to charges and time-varying currents. In Section 10.5 we introduced the concept of the impressed electric field, \( \mathbf{E}_i \). It was defined as a field equivalent to nonelectric forces acting on electric charges. Therefore the expression \( \mathbf{J} \cdot \mathbf{E}_i \) is the power of impressed (external) distributed sources per unit volume.

With this in mind, consider Maxwell’s differential equations (19.16) and (19.17), which we repeat for convenience:

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (19.35 = 19.16) \]

\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \quad (19.36 = 19.17) \]

To obtain a power expression that must be satisfied by an electromagnetic field, we must combine both of these equations because both must simultaneously be satisfied for a real field. Let us therefore multiply (find the dot product of) the first of these equations by \( \mathbf{H} \), the second by \(-\mathbf{E}\), and then add the two equations thus obtained. The result is

\[ \mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H} = -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \mathbf{J} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}. \quad (19.37) \]

Now, from vector analysis (see Appendix 2, No. 21)

\[ \mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H} = \nabla \cdot (\mathbf{E} \times \mathbf{H}). \quad (19.38) \]

If we assume the medium to be linear, we can write

\[ \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} = \frac{\partial}{\partial t} \left( \frac{1}{2} \mu \mathbf{H} \cdot \mathbf{H} \right) = \frac{\partial}{\partial t} \left( \frac{1}{2} \mu H^2 \right), \quad (19.39) \]

and

\[ \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \varepsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{\partial}{\partial t} \left( \frac{1}{2} \varepsilon E^2 \right), \quad (19.40) \]
For linear media, \( J = \sigma (E + E_i) \), so that \( E = (J/\sigma - E_i) \). If we substitute this expression of \( E \) into the term \( E \cdot J \) in Eq. (19.37), taking into account Eqs. (19.38) to (19.40), after simple manipulations Eq. (19.37) becomes

\[
E_i \cdot J = \frac{j^2}{\sigma} + \frac{\partial}{\partial t} \left( \frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right) + \nabla \cdot (E \times H). \tag{19.41}
\]

Let us multiply this equation by a volume element \( dv \) and integrate over an arbitrary volume \( v \) of the field. The last term of the equation thus obtained is a volume integral of the divergence of the vector \( (E \times H) \). By the use of the divergence theorem, this volume integral can be transformed into a surface integral over the surface \( S \) bounding the volume \( v \). So we finally obtain

\[
\int_v E_i \cdot J \, dv = \int_v \frac{j^2}{\sigma} \, dv + \frac{\partial}{\partial t} \int_v \left( \frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right) \, dv + \oint_S (E \times H) \cdot dS. \tag{19.42}
\]

(Poynting’s theorem)

This is Poynting’s theorem. It tells us about power balance inside a volume \( v \) of the electromagnetic field.

The term on the left represents the power of all the sources inside \( v \). The terms on the right show how this power is used. One part (represented by the first term) is transformed inside \( v \) into heat. The other part (represented by the second term) is used to change (increase if positive, decrease if negative) the energy localized in the electric and magnetic field inside \( v \). Because we consider a finite volume of the field, we need a term representing possible exchange of energy with the rest of the field, through the boundary of \( v \), that is, surface \( S \). According to Poynting, the last term on the right has precisely that meaning:

\[
\oint_S (E \times H) \cdot dS = \text{power transferred through } S \text{ to a region outside } S. \tag{19.43}
\]

This statement is also frequently considered as Poynting’s theorem.

According to Poynting’s theorem, the cross product \( (E \times H) \) can be interpreted as the power transferred by the electromagnetic field per unit area. The direction of the vector \( (E \times H) \) then shows the direction of transfer of energy through a surface perpendicular to it. The vector \( (E \times H) \) is referred to as the Poynting vector. We will designate it by \( P \) (calligraphic \( P \)): 
\[ \mathcal{P} = \mathbf{E} \times \mathbf{H} \quad (\text{W/m}^2). \quad (19.44) \]

(Definition of the Poynting vector)

The unit of Poynting's vector is W/m² (watts per square meter).

Poynting's theorem, as a mathematical expression of the law of conservation of energy in the electromagnetic field, is an extremely useful theorem. Note, however, that it is valid only for electromagnetic fields that are described simultaneously by the first and second of Maxwell's equations. The following example shows that in other cases Poynting's theorem does not make sense.

Example 19.4—Formal application of Poynting's theorem to crossed electrostatic and magnetostatic fields. Consider the system shown in Fig. 19.3. A charged parallel-plate capacitor and a permanent magnet are positioned so that their fields (electrostatic and magnetostatic) overlap. Consequently, considered formally, Poynting's vector in the figure is directed into the page. This could be interpreted as if energy is perpetually circulating through this region, and the only problem is how to capture it. This reasoning, however, is not correct. These electric and magnetic fields are not coupled (we can move the magnet, for example, without affecting the electric field of the capacitor). Combining the two fields in this case is like combining potatoes and oranges.

Example 19.5—Energy transfer through a coaxial cable. The cross section of a coaxial cable is sketched in Fig. 19.4. Assume that the voltage between the cable conductors is \( V \), and that there is a dc current in the cable of intensity \( I \), as indicated. It is a simple matter to conclude that the generator is connected in the direction toward the reader, and the load away from the reader. The Poynting vector is directed away from the reader. According to the interpretation of the Poynting vector, this means that energy is flowing through the cable away from the generator, as it should.

![Figure 19.3 Crossed electrostatic and magnetostatic fields](image)

Figure 19.3 Crossed electrostatic and magnetostatic fields
It is left as an exercise for the reader to prove that the flux of the Poynting vector through the cross section of the cable equals exactly $VI$. Note that the intensity of the Poynting vector is the largest at the inner conductor surface, which means that most of the power flows near that surface.

**Example 19.6—Poynting’s theorem in complex form.** Starting from the complex form of Maxwell’s equations, it is not difficult to obtain Poynting’s theorem in complex form. The principal difference of the derivation is that we start from the complex form of the first equation, from the *complex conjugate* form of the second equation, and multiply the first equation (find the dot product) with the *complex conjugate*, $\mathbf{H}^*$, of the vector $\mathbf{H}$. The rest of the derivation is quite similar to that given for Poynting’s theorem for arbitrary time dependence, and it is left as an exercise for the reader. The result is

$$
\int_{S} \mathbf{E} \cdot \mathbf{J^*} \, d\mathbf{v} = \int_{S} \frac{2}{\sigma} \, d\mathbf{v} + 2j\omega \int_{V} \left( \frac{1}{2} \mu \mathbf{H}^2 - \frac{1}{2} \varepsilon \mathbf{E}^2 \right) \, dv + \oint_{S} (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{S}. \quad (19.45)
$$

(Poynting’s theorem in complex form)

This is Poynting’s theorem in complex form. The vector

$$
\mathcal{P} = \mathbf{E} \times \mathbf{H}^* \quad \text{(W/m}^2\text{)} \quad (19.46)
$$

(The complex Poynting vector)

is known as the *complex Poynting vector*.

The equation expressing the Poynting theorem in complex form has a real and an imaginary part. It is left to the reader as an exercise to write these two parts of the equation and to discuss their meaning.

**Questions and problems:** Q19.25 to Q19.30, P19.5 to P19.9
19.7 The Generalized Definition of Conductors and Insulators

For linear media and time-harmonic variation of the fields, it is possible to clearly distinguish what a good conductor and a good insulator are. Let a time-harmonic electromagnetic field of angular frequency \( \omega \) exist in a medium of permittivity \( \epsilon \) and conductivity \( \sigma \). The second Maxwell’s equation in complex form becomes

\[
\nabla \times \mathbf{H} = (\sigma + j\omega\epsilon)\mathbf{E}.
\]

(19.47)

For a perfect dielectric, \( \sigma \) in this equation does not exist. For a very good conductor, displacement current is negligible, so the term \( j\omega\epsilon \) is missing. Thus, at a frequency \( f = \omega/(2\pi) \), we can define a good conductor by the inequality

\[
\sigma \gg \omega\epsilon \quad \text{(definition of a good conductor)},
\]

(19.48)

and a good insulator by the inequality

\[
\sigma \ll \omega\epsilon \quad \text{(definition of a good insulator)}.
\]

(19.49)

19.8 The Lorentz Potentials

In Chapter 4 we introduced the concept of the electric scalar potential. This is just one in a family of potentials used in the analysis of electromagnetic fields. A potential is an auxiliary scalar or vector function, which is usually easier to calculate than the field vectors themselves, and from which the field vectors are obtained in some simple manner, usually by differentiation.

We will introduce here a pair of potentials that seem to be used most often in electromagnetic field analysis. One of these is the generalized scalar potential we already know. The other is a vector function, known as the **magnetic vector potential**. The specific pair of potentials we will now derive are known as the **Lorentz potentials**. For reasons to become apparent later, they are also known as the **retarded potentials**.

Note first that \( \nabla \cdot (\nabla \times \mathbf{A}) = 0 \) for any vector function \( \mathbf{A} \) (see Appendix 2, No. 24). Since \( \nabla \cdot \mathbf{B} = 0 \), it follows that it is always possible to express the magnetic flux density vector \( \mathbf{B} \) as

\[
\mathbf{B} = \nabla \times \mathbf{A}.
\]

(19.50)

(Definition of magnetic vector potential)

The vector function \( \mathbf{A} \) is known as the **magnetic vector potential**.

If the expression for \( \mathbf{B} \) in Eq. (19.50) is introduced into the first Maxwell’s equation, we obtain

\[
\nabla \times \mathbf{E} = -\frac{\partial}{\partial t} (\nabla \times \mathbf{A}).
\]

(19.51)
This means that $\nabla \times (E + \partial A/\partial t) = 0$. Now, we know that $\nabla \times (\nabla V) = 0$ always [see Appendix 1, Eq. (A1.50)]. Therefore Eq. (19.51) implies that $(E + \partial A/\partial t) = -\nabla V$, and not zero. (The negative gradient is used for convenience.) Thus the electric field strength can be expressed as

$$E = -\nabla V - \frac{\partial A}{\partial t}. \quad (19.52)$$

(Electric field strength in terms of retarded potentials)

Evidently, for time-invariant fields $V$ becomes the electric scalar potential we know. Therefore we retain the same name for $V$ in this case, where $V$ is an arbitrary function of time.

So we have two equations, (19.50) and (19.52), from which we can easily calculate vectors $E$ and $B$, provided we know the two potentials, $V$ and $A$. For obtaining Eqs. (19.50) and (19.52) we used the first and the fourth Maxwell’s equations. For determining these potentials in terms of the field sources, $\rho$ and $J$, we therefore make use of the other two Maxwell’s equations.

Let us assume that the medium is linear and homogeneous, of permittivity $\varepsilon$ and permeability $\mu$. Then, substituting Eqs. (19.50) and (19.52) into the second and third Maxwell’s equation, we obtain, respectively,

$$\nabla \times (\nabla \times A) = \mu J - \varepsilon \mu \frac{\partial}{\partial t}(\nabla V) - \varepsilon \mu \frac{\partial^2 A}{\partial t^2}, \quad (19.53)$$

and

$$\nabla \cdot (\nabla V) = \nabla^2 V = -\frac{\rho}{\varepsilon} - \nabla \cdot \frac{\partial A}{\partial t}. \quad (19.54)$$

Since $\nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) - \nabla^2 A$ [see Appendix 1, Eq. (A1.37)], Eq. (19.53) becomes

$$\nabla^2 A = -\mu J + \varepsilon \mu \nabla \frac{\partial V}{\partial t} + \varepsilon \mu \frac{\partial^2 A}{\partial t^2} + \nabla (\nabla \cdot A). \quad (19.55)$$

There is a theorem in vector analysis called the Helmholtz theorem. It says that a vector function is uniquely defined if its curl and divergence are known at every point in space. We already know what the curl of $A$ is ($\nabla \times A = B$), so we need to define its divergence in order that it be unique. Because only $\nabla \times A$ matters ($B = \nabla \times A$), we can define $\nabla \cdot A$ in an infinite number of ways, resulting in an infinite number of pairs of potentials $A$ and $V$. Having this freedom of choice, it is wise to adopt $\nabla \cdot A$ so that we can solve Eqs. (19.54) and (19.55) most easily.

It is a simple matter to conclude that if we adopt the Lorentz condition for $\nabla \cdot A$,

$$\nabla \cdot A = -\varepsilon \mu \frac{\partial V}{\partial t}, \quad (19.56)$$

(The Lorentz condition)
Eqs. (19.54) and (19.55) take the simplest possible forms, each becoming a partial differential equation in a single unknown, \( V \) in the first case and \( A \) in the second case:

\[
\nabla^2 V - \epsilon \mu \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon},
\]

(19.57)

\[
\nabla^2 A - \epsilon \mu \frac{\partial^2 A}{\partial t^2} = -\mu J.
\]

(19.58)

Because the \( x \) component of the last equation in a rectangular coordinate system is given by [see Appendix 1, Eq. (A1.39)]

\[
\nabla^2 A_x - \epsilon \mu \frac{\partial^2 A_x}{\partial t^2} = -\mu J_x,
\]

(19.59)

and similarly for the \( y \) and \( z \) components, we need to solve only Eq. (19.57). The solution of Eq. (19.58) will then be obtained as a vector sum of analogous solutions for the vector components of \( A \).

Solving Eq. (19.57) is not simple and does not add anything to the understanding of the final result. We therefore give only the final result:

\[
V(r, t) = \frac{1}{4\pi\epsilon} \int_{\nu'} \frac{\rho(r', t - R/c)}{R} d\nu' \quad c = \frac{1}{\sqrt{\epsilon\mu}}.
\]

(19.60)

So the solution of Eq. (19.58) is

\[
A(r, t) = \frac{\mu}{4\pi} \int_{\nu'} \frac{J(r', t - R/c)}{R} d\nu' \quad c = \frac{1}{\sqrt{\epsilon\mu}}.
\]

(19.61)

These are the Lorentz potentials. The meaning of \( r, r' \), and \( R \) is illustrated in Fig. 19.5.

![Figure 19.5 Explanation of symbols in Eqs. (19.60) and (19.61)](image-url)
We stress again that the entire derivation, and therefore also Eqs. (19.60) and (19.61), is valid only for homogeneous and linear media.

What is the physical meaning of the expressions for the potentials in Eqs. (19.60) and (19.61)? Say there is an elemental source at a point \( P' \) whose position vector is \( \mathbf{r}' \), as in Fig. 19.5. We are observing the fields due to this source at a point \( P \) defined by the position vector \( \mathbf{r} \). The magnitude of the field at point \( P \) at a time \( t \) is not the one that the source produces at time \( t \), but at an earlier time, \( t - \frac{R}{c} \). In other words, in a homogeneous dielectric of permittivity \( \varepsilon \) and permeability \( \mu \), the fields propagate with a finite velocity, \( c = 1/\sqrt{\varepsilon \mu} \), that is, they are retarded in reaching the field point. For this reason, the Lorentz potentials are often termed the retarded potentials.

In the case of a vacuum (\( \varepsilon = \varepsilon_0, \mu = \mu_0 \)), the velocity \( c \) of propagation of the potentials becomes exactly the speed of light in a vacuum, \( c_0 \), a calculation left as an exercise for the reader.

**Example 19.7—Retarded potentials in complex (phasor) form.** Very often, sources of an electromagnetic field are time-harmonic. In that case the retarded potentials can be written without explicit time dependence. The procedure for obtaining the complex potentials is simple—we just assume the sources, \( \rho \) and \( \mathbf{J} \), and the potentials to vary following the law \( e^{j\omega t} \). So we obtain

\[
V(\mathbf{r}) = \frac{1}{4\pi \varepsilon} \int_{V'} \frac{\rho(\mathbf{r}')e^{-j\omega R/c}}{R} \, d\mathbf{r}', \quad c = \frac{1}{\sqrt{\varepsilon \mu}},
\]

(19.62) *(Complex retarded scalar potential)*

and

\[
A(\mathbf{r}) = \frac{\mu}{4\pi} \int_{V'} \frac{\mathbf{J}(\mathbf{r}')e^{-j\omega R/c}}{R} \, d\mathbf{r}', \quad c = \frac{1}{\sqrt{\varepsilon \mu}}.
\]

(19.63) *(Complex retarded vector potential)*

**Example 19.8—Definition of quasi-static fields.** It is interesting that for time-harmonic fields it is possible to inspect whether a field in a system can be considered practically as a static field (or a quasi-static field), or not.

From Eqs. (19.62) and (19.63) we see that the retardation can be neglected provided that the largest dimension of the field domain we consider, \( d_{\text{max}} \), is determined by the inequality \( \omega d_{\text{max}}/c \ll 1 \), or

\[
d_{\text{max}} \ll \frac{c}{\omega} = \frac{1}{\omega \sqrt{\varepsilon \mu}} \quad \text{(the condition for quasi-static fields)}.
\]

**Questions and problems:** Q19.31 to Q19.36, P19.10 to P19.13

### 19.9 Chapter Summary

1. The general equations of the electromagnetic field, known as Maxwell’s equations, are as fundamental in electromagnetic field theory as are Newton’s laws in mechanics.
2. Some general consequences of Maxwell’s equations include general boundary conditions, the Poynting theorem (the theorem on the transfer of energy by the electromagnetic field), and the possibility of making a clear distinction between conductors and insulators for time-harmonic fields.

3. Field vectors can be expressed in terms of auxiliary functions, called potentials.

4. The expressions for potentials indicate that the speed of electromagnetic disturbances in a vacuum is that of light.

5. For sinusoidal field variation, the expressions for the potentials in complex form enables us to define the dimensions of systems in which fields can be considered approximately as static (quasi-static fields).

QUESTIONS

Q19.1. Why (and when) is it allowed to move the time derivative in Eq. (19.1) to act on $D$ only, and thus obtain Eq. (19.2)?

Q19.2. Does Eq. (19.5) tell us that a time-varying magnetic field is the source of a time-varying electric field? Explain.

Q19.3. Why would an electric field inside a perfect conductor produce a current of infinite density? Would such a current be physically possible? Explain.

Q19.4. Why are surface currents possible on surfaces of perfect conductors, when a nonzero tangential electric field there is not possible? Is this a current of finite volume density?

Q19.5. Write the full set of Maxwell’s equations in differential form for the special case of a static electric field, assuming that the dielectric is linear, but inhomogeneous.

Q19.6. Write the full set of Maxwell’s equations in differential form for the special case of a static electric field produced by the charges on a set of conducting bodies situated in a vacuum.

Q19.7. Write the full set of Maxwell’s equations in differential form for the special case of a steady current flow in a homogeneous conductor of conductivity $\sigma$, with no impressed electric field.

Q19.8. Write the full set of Maxwell’s equations in differential form for the special case of a steady current flow in an inhomogeneous poor dielectric, with impressed electric field $E$ present.

Q19.9. Write the full set of Maxwell’s equations in differential form for the special case of a time-constant magnetic field in a linear medium of permeability $\mu$, produced by a steady current flow.

Q19.10. Write the full set of Maxwell’s equations in differential form for the special case of a time-constant magnetic field, produced by a permanent magnet of magnetization $M$ (a function of position).

Q19.11. Write the full set of Maxwell’s equations in differential form for the special case of a time-constant magnetic field produced by both steady currents and magnetized matter, if the medium is not linear.

Q19.12. Write the full set of Maxwell’s equations in differential form for the special case of a quasi-static electromagnetic field, produced by quasi-static currents in nonferromagnetic conductors.
Q19.13. Write Maxwell’s equations in differential form for an arbitrary electromagnetic field in a vacuum, no free charges being present.

Q19.14. Write Maxwell’s equations for an arbitrary electromagnetic field in a homogeneous perfect dielectric of permittivity $\epsilon$ and permeability $\mu$.

Q19.15. Write the full set of differential Maxwell’s equations in scalar form in the rectangular coordinate system. Note that eight simultaneous, partial differential equations result. Write these equations neatly and save them for future reference.

Q19.16. Repeat question Q19.15 for the cylindrical coordinate system.

Q19.17. Repeat question Q19.15 for the spherical coordinate system.

Q19.18. Write differential Maxwell’s equations in scalar form for the particular case of an electromagnetic field in a vacuum ($\mathbf{J} = 0, \rho = 0$), if the field vectors are only functions of the cartesian coordinate $z$ and of time $t$.

Q19.19. Write differential Maxwell’s equations in scalar form for a good conductor, for the particular case of an axially symmetrical system with dependence of the field vectors only on the cylindrical coordinate $\tau$ and time $t$. Assume that $\mathbf{J} = j_0 \mathbf{u}_z$ and $\rho = 0$.

Q19.20. Repeat question Q19.19 for $\mathbf{B} = B_z \mathbf{u}_z$.

Q19.21. Write differential Maxwell’s equations in complex form for an arbitrary electromagnetic field in a very good conductor, of conductivity $\sigma$ and permeability $\mu$.

Q19.22. Write differential Maxwell’s equations in complex form for a quasi-static electromagnetic field.

Q19.23. Write differential Maxwell’s equations in complex form for an arbitrary electromagnetic field in a perfect dielectric of permittivity $\epsilon$ and permeability $\mu$, no free charges being present.

Q19.24. Write the most general integral Maxwell’s equations in complex form.

Q19.25. The current intensity through a resistor of resistance $R$ is $I$. What is the flux of the Poynting vector through any closed surface enclosing the resistor?

Q19.26. A capacitor, of capacitance $C$, is charged with a charge $Q$. What is the flux of the Poynting vector through any surface enclosing the capacitor, if the charge $Q$ (1) is constant in time, or (2) varies in time as $Q = Q_m \cos \omega t$?

Q19.27. A coil, of inductance $L$, carries a current $i(t)$. What is the flux of the Poynting vector through any surface enclosing the coil?

Q19.28. A dc generator of emf $E$ is open-circuited. What is the flux of the Poynting vector through any surface enclosing the generator?

Q19.29. Repeat question Q19.28 assuming that a current $i(t)$ flows through the generator, and its internal resistance is $R$.

Q19.30. What is the time-average value of the Poynting vector, if complex rms values are known for the electric and magnetic field strength, $\mathbf{E}$ and $\mathbf{H}$?

Q19.31. The largest dimension of a coil at a very high frequency is on the order of $(\omega \sqrt{\epsilon \mu})^{-1}$. Is it possible at such high frequencies to define the inductance of the coil in the same way as in a quasi-static case? Explain.

Q19.32. The length of a long 60-Hz power transmission line is equal to $0.5(\omega \sqrt{\epsilon_0 \mu_0})^{-1}$. Is this a quasi-static system? What is the length of the line?

Q19.33. A parallel-plate capacitor has plates of linear dimensions comparable with $(\omega \sqrt{\epsilon \mu})^{-1}$, where $\omega$ is the operating angular frequency. Is it possible to determine the capacitance of such a capacitor in the same way as in the static and quasi-static case? Explain.
Q19.34. An electric circuit operates at a high frequency $f$. The largest linear dimension of the circuit is $2(\omega_0/e\mu)^{-1}$. Are Kirchhoff’s laws applicable in this case for analyzing the circuit? Explain.

Q19.35. Write the Lorentz condition in complex form.

Q19.36. A current pulse of duration $\Delta t = 10^{-9}$ s was excited in a small wire loop. After how many $\Delta t$'s is the magnetic and induced electric field of this pulse going to be detected at a point $r = 10$ m from the loop?

## PROBLEMS

P19.1. A current $i(t) = I_m \cos \omega t$ flows through the leads of a parallel-plate capacitor of plate area $S$ and distance between them $d$. If the permittivity of the dielectric of the capacitor is $\epsilon$, prove that the displacement current through the capacitor dielectric is exactly $i(t)$. Ignore fringing effects.

P19.2. A spherically symmetrical charge distribution disperses under the influence of mutually repulsive forces. Suppose that the charge density $\rho(r, t)$, as a function of the distance $r$ from the center of symmetry and of time, is known. Prove that the total current density at any point is zero.

P19.3. Determine the magnetic field as a function of time for the dispersing charge distribution in problem P19.2.

P19.4. Small-scale models are used often in engineering practice, including electrical engineering. Starting from differential Maxwell’s equations for a linear medium, derive the necessary conditions for the electromagnetic field in a small-scale model to be similar to the field in a real, $n$ times larger model. (These conditions are usually referred to as the conditions of the *electrodynamic similitude*.) (Hints: (1) Write the first two differential Maxwell’s equations for the full-scale system, and for the model. (2) Note that the coordinates in the latter are $n$ times smaller, and find the conditions under which, in spite of that, the two sets of equations will be the same.)

P19.5. A lossless coaxial cable, of conductor radii $a$ and $b$, carries a steady current of intensity $I$. The potential difference between the cable conductors is $V$. Prove that the flux of the Poynting vector through a cross section of the cable is $VI$, using the known expressions for vectors $E$ and $H$ in the cable. Sketch the dependence of the magnitude of the Poynting vector on the distance $r$ from the cable axis, where $a < r < b$.

P19.6. Repeat problem P19.5 for an air stripline with strips of width $a$ that are a distance $d$ apart, if the current in the strips is $I$ and voltage between them $V$. Neglect the edge effects.

P19.7. The stripline from the preceding problem is connected to a sinusoidal generator of emf $E$ and angular frequency $\omega$. The other end of the line is connected to a capacitor of capacitance $C$. Apply Poynting’s theorem in complex form to a closed surface enclosing (1) the generator, or (2) the capacitor.

P19.8. Repeat problem P19.7 assuming that the load is an inductor of inductance $L$, instead of a capacitor.

P19.9. Repeat problem P19.7, assuming that the line is a lossless coaxial line of conductor radii $a$ and $b$.

P19.10. Derive Eqs. (19.57) and (19.58) from Eqs. (19.54), (19.55), and (19.56).

P19.11. Derive the retarded potentials in Eqs. (19.62) and (19.63) from Eqs. (19.60) and (19.61).
P19.12. Suppose a system is regarded as approximately quasi-static if its largest dimension $d$ satisfies the inequality $d \omega \sqrt{\epsilon \mu} \leq 0.1$. Determine the largest value of $d$ thus defined for the electrodynamic systems in a vacuum if the frequency of the generators is (1) 60 Hz, (2) 10 MHz, or (3) 10 GHz.

P19.13. Compare the rms values of vectors $\mathbf{J}$ and $\partial \mathbf{D}/\partial t$ in copper, seawater, and wet ground, for frequencies $f$ of (1) 60 Hz, (2) 10 kHz, (3) 100 MHz, or (4) 10 GHz. For copper, assume $\epsilon = \epsilon_0$, $\sigma = 56 \cdot 10^6$ S/m. For seawater, adopt $\epsilon = 10 \epsilon_0$, $\sigma = 4$ S/m, and for the ground $\epsilon = 10 \epsilon_0$ and $\sigma = 10^{-2}$ S/m.