A general analysis method for feedback circuits

\[ V_e = V(+) - V(-) \]

Solve for closed-loop response

\[ A_{cl} = \frac{V_o}{V_i} \]

\[ V_e = V(+) - V(-) \quad \text{"error signal"} \]

\[ V_o = (A V_e) \cdot G_3 = AG_3 V_e \]

\[ V(+) = G_1 V_i \]

\[ V(-) = G_2 V_o \]

\[ V_o = AG_3 \left( G_1 V_i - G_2 V_o \right) \]

\[ V_o \left( 1 + AG_2 G_3 \right) = AG_1 G_3 V_i \]

\[ A_{cl} = \frac{V_o}{V_i} = \frac{AG_1 G_3}{1 + AG_2 G_3} \]
Rearrange:

\[ Acl = \frac{V_o}{V_i} = \frac{G_1}{G_2} \frac{A G_2 G_3}{1 + A G_2 G_3} \]

\[ Acl = (Acl)_{ideal} \frac{T}{1 + T} \]

where

\[ (Acl)_{ideal} = \frac{V_o}{V_i} \Bigg|_{T \to \infty} = \left( \frac{V_o}{V_i} \right) \Bigg|_{V_e = 0} = \frac{G_1}{G_2} \]

is the ideal closed-loop response when the error signal \( V_e \) is zero.

and \( T = A G_2 G_3 \) is the loop gain.

The general feedback circuit analysis:

1. Identify the feedback loop and the error \( V_e \) and verify that the feedback is negative.
2. Find \( (Acl)_{ideal} = \frac{V_o}{V_i} \Bigg|_{V_e = 0} \) under the condition that the error is zero, i.e., under the condition that \( T \to \infty \).
3. Find \( T \).
4. \( Acl = (Acl)_{ideal} \frac{T}{1 + T} \).
\[ T = \left. \frac{V_o}{V_x} \right|_{V_i = 0} = \frac{V_d}{V_x} \frac{V_e}{V(-)} \frac{V(-)}{V_0} \frac{V_0}{V_x} = \]

\[ = (-A)(-1) G_2 G_3 = AG_2 G_3 \]

(1) null the input, \( V_i = 0 \)

(2) insert a test source \( V_{test} \) in series with feedback loop. A convenient point is the output of the amplifier \( A \)

(3) find \( T = \frac{V_d}{V_x} \)
Examples:

1. $v_e = v(+) - v(-)$

If $v_e = 0$, then $\frac{v_o}{v_i} = (Acl)\text{ideal} = -\frac{R_b}{R_a}$

Find the loop gain $T$:

$$T(s) = \frac{v_y}{v_x} = \frac{v_y}{v(-)} \frac{v(-)}{v_x} = A(s) \frac{R_a}{R_a + R_b}$$

So,

$$Acl(s) = -\frac{R_b}{R_a} \frac{A(s) \frac{R_a}{R_a + R_b}}{1 + A(s) \frac{R_a}{R_a + R_b}}$$
same as #1, except $R_{out} \neq 0$, and $C_L$ is a capacitive load.

$$(A_{in})_{ideal} = \frac{V_o}{V_i} \bigg|_{V(+)=V(-)} = -\frac{R_b}{R_a}, \text{ same as in #1.}$$

Find $T(s)$

$$T(s) = \frac{V_o}{V_x} = \frac{V_x}{V(-)} \frac{V(-)}{V_o} \frac{V_o}{V_x}$$

$$= A(s) \frac{R_a}{R_a + R_b} \frac{1}{V_{CL}} \frac{1}{R_{out} + \frac{1}{V_{CL}}}$$

$$= \frac{1}{A(s)} \frac{R_a}{R_a + R_b} \frac{1}{1 + S C_L R_{out}}$$

if $R_b \gg R_{out}$

$$T(s) = A(s) \cdot \frac{R_a}{R_a + R_b} \cdot \frac{1}{1 + S C_L R_{out}}$$

$$\text{and} \quad A_{out}(s) = -\frac{R_b}{R_a} \frac{T(s)}{1 + T(s)}$$
\[(Au)_{\text{ideal}} = 1\]

Find \(T(s)\):

\[
T(s) = \frac{V_o}{V_x} \bigg| V_i = 0 = \frac{V_y}{V_x} \frac{V(+) - V(-)}{V(+)} \frac{V_o}{V_x} =
\]

\[
= (-A(s)) \left( \frac{1}{(1 + g_w R)} \right)
\]

\[
T(s) = g_w R A(s)
\]

\[
\frac{V_o}{V_i} = 1 - \frac{g_w R A(s)}{1 + g_w R A(s)}
\]
Effects of feedback on the frequency response

Consider the case when \( T(s) \) is a single-pole function and \((A_{cl})_{\text{ideal}}\) is a constant:

\[
(A_{cl})_{\text{ideal}} = \text{const.}
\]

\[
T(s) = \frac{T(0)}{1 + \frac{s}{\omega_p}}
\]

\[
A_{cl}(s) = (A_{cl})_{\text{ideal}} \frac{T(s)}{1 + T(s)} = \frac{T(0)}{1 + \frac{s}{\omega_p}}
\]

\[
A_{cl}(s) = (A_{cl})_{\text{ideal}} \frac{T(0)}{1 + T(0)} \frac{1}{1 + \frac{s}{\omega_p (1 + T(0))}}
\]

\[ T(0) \gg 1, \]

\[
A_{cl}(s) \approx (A_{cl})_{\text{ideal}} \uparrow \text{low-freq. gain} \uparrow \text{CL pole frequency}
\]

\[
\text{Bandwidth is}
\]

\[
BW_{cl} = T(0) \omega_p = f_c = \text{"UNITY-GAIN FREQ. OF THE LOOP-GAIN"}
\]

\[ = \text{"CROSS-OVER FREQUENCY"} \]
$f_c = T(0) f_p = \text{CROSS-OVER FREQUENCY}$

$\text{BW of the closed-loop application is}$

$\text{equal to the cross-over frequency of the}\$

$\text{loop gain.}$

Examples of finding closed-loop BW:

- noninverting amplifier
- inverting amplifier
- voltage regulator
Single-pole \( T(s) = \frac{T(0)}{1 + \frac{s}{\omega_p}} \),

\[ T(0) \omega_p = f_c = \text{"cross-over frequency"} \]

= unity-gain frequency of the loop gain

\[ |T(j\omega_c)| = 1 \]

Magnitude and phase response of the loop-gain:

\[ 20 \log |T(j\omega)| \]

\[ \phi T(j\omega) \]

\[ -90^\circ \]

\[ -180^\circ \]

achal response with HF poles and zeros

actual response with HF poles and zeros

\[ \phi T(j\omega) = -180^\circ = \text{positive feedback at this frequency!} \]

\[ |T(j\omega)| > 1 \rightarrow \text{oscillations at } f_x \text{ grow, INSTABILITY} \]

\[ |T(j\omega)| < 1 \rightarrow \text{oscillations decay, STABILITY} \]
$|T(j\omega)| = 1$

PM = $\gamma_M = \text{phase margin}$

$= 180^\circ + \angle T(j\omega_c)$

$\gamma_\text{M} > 0 \implies \text{STABLE}$

$\gamma_\text{M} < 0 \implies \text{UNSTABLE}$

STABILITY TEST
FINDING PHASE MARGIN $\varphi_M = PM$

1. Find loop gain $T(s)$

2. Find cross-over frequency $f_c = \text{unity gain frequency of the loop gain}$,

   \[ |T(j\omega_c)| = 1 \quad \text{or} \quad 20 \log |T(j\omega_c)| = 0 \]

3. $\varphi_M = PM = 180^\circ + \angle T(j\omega_c)$

---

Step (2) is greatly simplified if straight-line approximations to $20 \log |T(j\omega)|$ are used instead of the exact magnitude response.

**Example**

Consider a 2-pole $T(s)$,

\[ T(s) = \frac{T(0)}{(1 + \frac{s}{\omega_p1})(1 + \frac{s}{\omega_p2})} \]

There are 2 cases:

# 1 $f_{p2} > T(0) f_{p1}$

# 2 $f_{p2} < T(0) f_{p1}$

The Bode plots of the magnitude response of the loop gain for the two cases are:
\[
\begin{align*}
20 \log |T(j \omega_c)| &= 20 \log |T(0)| - 20 \log \sqrt{1 + \left(\frac{\omega_c}{\omega_{p1}}\right)^2} - \\
&\quad - 20 \log \sqrt{1 + \left(\frac{\omega_c}{\omega_{p2}}\right)^2} = 0
\end{align*}
\]

\[
20 \log |T(j \omega_c)| \approx 20 \log |T(0)| - 20 \log \frac{\omega_c}{\omega_{p1}} = 0
\]

\[
\Rightarrow \quad f_c \approx T(0) f_{p1}
\]

\[
\Phi_M = \phi_M = 180 - \arctg \frac{f_c}{f_{p1}} - \arctg \frac{f_c}{f_{p2}}
\]

\[
= 180 - \arctg \left(\frac{T(0)}{f_{p1}}\right) - \arctg \frac{T(0) f_{p1}}{f_{p2}}
\]

\[
\approx 90^\circ \quad \text{if} \quad T(0) \gg 1
\]

\[
\Phi_M = \phi_M \approx 90^\circ - \arctg \frac{T(0) f_{p1}}{f_{p2}}
\]

\[
i_f \quad T(0) f_{p1} < f_{p2}
\]
\[ \begin{align*}
20 \log |T(j\omega)| & \approx 20 \log |T(0)| - 20 \log \frac{f_{p2}}{f_{p1}} - 40 \log \frac{f_c}{f_{p2}} = 0 \\
T(0) &= \frac{f_{p2}}{f_{p1}}\\
T(0) &\approx \frac{f_{p2}}{f_{p1}} \cdot \frac{f_c^2}{f_{p2}} \\
&\Rightarrow f_c \approx \sqrt{T(0)f_{p1}f_{p2}} \\
PM = \gamma_M &\approx 180 - \arctg \frac{f_c}{f_{p1}} - \arctg \frac{f_c}{f_{p2}} \\
&= 180 - \arctg \sqrt{\frac{T(0)f_{p2}}{f_{p1}}} - \arctg \sqrt{\frac{T(0)f_{p1}}{f_{p2}}} \\
PM = \gamma_M &\approx 90^\circ - \arctg \sqrt{\frac{T(0)f_{p1}}{f_{p2}}} \\
&\text{if } T(0)f_{p1} > f_{p2}
\end{align*} \]
The results in case #1 are the same for $T(s)$ with arbitrary # of poles/zeros, as long as $T(0)f_{p1} < \text{frequencies of all other poles and zeros}$.

For $T(s) = T(0) \frac{(1 + \frac{s}{\omega_{p1}})(1 + \frac{s}{\omega_{p2}}) \ldots}{(1 + \frac{s}{\omega_{m}})(1 + \frac{s}{\omega_{m}}) \ldots}$

$PM = \gamma_m = 180^\circ - \arctg \frac{f_c}{f_{p1}} - \arctg \frac{f_c}{f_{p2}} - \ldots \ (\text{poles})$

$+ \arctg \frac{f_c}{f_{z1}} + \arctg \frac{f_c}{f_{z2}} + \ldots \ (\text{zeros})$

General PM calculation for a given $T(s)$ with cross-over frequency $f_c$. 

Notes:
How much Phase Margin is Needed?

Link between $PM = PM$ and the closed-loop responses. See Text pages 236-239.

Step response of a closed-loop application:

1. $PM < \varnothing$: instability, oscillations continue forever
2. $0 < PM < 76^\circ$: stable response with overshoot of $p\%$, and oscillations that decay in time
3. $PM > 76^\circ$: stable response with no overshoot

See Table 5.1 in the Text:

<table>
<thead>
<tr>
<th>$PM$</th>
<th>$p%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\leq \varnothing$</td>
<td>100%</td>
</tr>
<tr>
<td>45°</td>
<td>16%</td>
</tr>
<tr>
<td>55°</td>
<td>13%</td>
</tr>
<tr>
<td>60°</td>
<td>9%</td>
</tr>
<tr>
<td>65°</td>
<td>5%</td>
</tr>
<tr>
<td>70°</td>
<td>1%</td>
</tr>
<tr>
<td>$&gt;76^\circ$</td>
<td>0%</td>
</tr>
</tbody>
</table>
Techniques for shaping $T(s)$ so that

PM $\geq$ minimum specified value, i.e.,

(45°, 60°, 75°, depending on the application)

$f_c$ as high as possible
(because $(BW)_{CL} \approx \frac{1}{f_c}$)

Standard, dominant-pole op-amp compensation

Use $C_c$ large enough so that

$A(0) f_{p1} < \text{high-frequency poles and zeros.}$

Recall: $A(0) = \frac{g_m R_1 g_m R_2}{1 + \frac{g_m R_1 R_2 C_c}{2 \pi}}$

$\frac{1}{f_{p1}} = \frac{1}{2 \pi g_m R_1 R_2 C_c}$

$A(0) f_{p1} = f_{na} = GBW = \frac{g_m}{2 \pi C_c}$

Condition ⚫ guarantees $PM > 45°$ in all closed-loop applications with resistive feedback, including $\overline{V_i} \rightarrow \overline{V_o}$ where $T(s) = A(s)$. 
In this part of the notes, we summarize the notation and the main results related to effects of feedback on the frequency response, including: gain-bandwidth trade-offs, stability limits, and compensation techniques.

<table>
<thead>
<tr>
<th>SYMBOL(S)</th>
<th>DEFINITION</th>
<th>NOTES</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_{-3dB}, w_{BW}$</td>
<td>$</td>
<td>A(jw_{BW})</td>
</tr>
<tr>
<td>$BW, f_{-3dB}, f_{BW}$</td>
<td>$BW = \frac{w_{BW}}{2\pi}$</td>
<td>Bandwidth in Hz.</td>
</tr>
<tr>
<td>$A(s), A_{OL}(s)$</td>
<td>$A(s) = \frac{v}{v(+) - v(-)}$</td>
<td>Op-amp open-loop transfer function, or transfer function of any amplifier.</td>
</tr>
<tr>
<td>$A(jw), A_{OL}(jw)$</td>
<td>$A(jw) = A(s)</td>
<td>_{s \rightarrow jw}$</td>
</tr>
<tr>
<td>$A(0), A_{OL}(0), A_{o}$</td>
<td>$A(0) = A(jw)</td>
<td>_{w=0}$</td>
</tr>
<tr>
<td>$20 \log</td>
<td>A(jw)</td>
<td>$</td>
</tr>
<tr>
<td>$f_{p1}, f_{1}$</td>
<td></td>
<td>Frequency of the low-frequency pole, dominant-pole frequency.</td>
</tr>
<tr>
<td>$f_{u}$</td>
<td>$</td>
<td>A(jwf_{u})</td>
</tr>
<tr>
<td>GBW</td>
<td>$GBW = f_{u}$</td>
<td>Op-amp gain-bandwidth product, equal to the op-amp unity-gain frequency.</td>
</tr>
<tr>
<td>SR</td>
<td>$SR = \frac{</td>
<td>dv_o}{dt}</td>
</tr>
</tbody>
</table>

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Department of Electrical and Computer Engineering
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<thead>
<tr>
<th>SYMBOL(S)</th>
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</thead>
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<tr>
<td>$T(s)$, LG</td>
<td>$T(s) = \frac{v_o}{v_r}</td>
<td>_{i=0}$</td>
</tr>
<tr>
<td>$T(jw)$</td>
<td>$T(jw) = T(s)</td>
<td>_{s\rightarrow jw}$</td>
</tr>
<tr>
<td>$T(0)$</td>
<td>$T(0) = T(jw)</td>
<td>_{w=0}$</td>
</tr>
<tr>
<td>$20 \log</td>
<td>T(jw)</td>
<td>$</td>
</tr>
<tr>
<td>$&lt; T(jw)$</td>
<td>$&lt; T(jw) = \arctan \frac{\text{Im}(T(jw))}{\text{Re}(T(jw))}$</td>
<td>Loop gain phase response.</td>
</tr>
<tr>
<td>$f_c, f_t$</td>
<td>$</td>
<td>T(jw_c)</td>
</tr>
<tr>
<td>PM, $\varphi_M$, $\phi_M$</td>
<td>PM = $180^\circ + &lt; T(jw_c)$</td>
<td>Phase margin (of the loop gain $T(jw)$).</td>
</tr>
<tr>
<td>$A_{CL}(s)$</td>
<td>Closed-loop transfer function of a feedback circuit application.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$A_{CL}(s) = (A_{CL})_{\text{ideal}} \left{ \frac{T(s)}{1+T(s)} \right}$</td>
<td></td>
</tr>
<tr>
<td>$(A_{CL})_{\text{ideal}}$</td>
<td>$(A_{CL})<em>{\text{ideal}} = A</em>{CL}(s)</td>
<td>_{T\rightarrow \infty}$</td>
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<td>$A_{CL}(jw)$</td>
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<tr>
<td>$20 \log</td>
<td>A_{CL}(jw)</td>
<td>$</td>
</tr>
<tr>
<td>$(BW)_{CL}$, BW</td>
<td></td>
<td>Closed-loop bandwidth in Hz of a feedback circuit. $(BW)_{CL} \approx f_c$, for a feedback circuit with large phase margin PM (for PM &gt; 45°).</td>
</tr>
</tbody>
</table>