We find $H_z$ by separation of variables exactly as was done in section 5.5 for $E_z$, up to eqn. (5.71):

$$H_z(\rho, \phi) = [C_c \cos \nu \phi + C_s \sin \nu \phi] J_\nu(k_c \rho)$$

The boundary conditions give

$$\frac{\partial H_z}{\partial \phi} \bigg|_{\phi=0} = 0 \quad \Rightarrow \quad C_s = 0$$

$$\frac{\partial H_z}{\partial \phi} \bigg|_{\phi=\frac{3\pi}{2}} = 0 \quad \Rightarrow \quad \sin \left( \frac{3\pi}{2} \right) = 0 \quad \Rightarrow \quad \nu = \frac{2}{3} l; \quad l = 0, 1, 2, \ldots$$

Note that the periodicity condition cannot be applied here, because the differential equation for $H_z$ does not hold continuously as $\phi$ is increased by $2\pi$—it is not valid in $\frac{3\pi}{2} < \phi < 2\pi$. Hence,

$$H_z(\rho, \phi) = C_c \cos \frac{2l\phi}{3} J_{\frac{2l}{3}}(k_c \rho)$$

The final portion of the boundary condition is

$$\frac{\partial H_z}{\partial \rho} \bigg|_{\rho=a} = 0 \quad \Rightarrow \quad k_c a = j'_{2l/3,m}; \quad m = 1, 2, \ldots$$

where $j'_{2l/3,m}$ is the $m$th root of the derivative of the Bessel function $J_{2l/3}$. For $l = a$ a multiple of 3, these roots can be found in Appendix C; for other values of $l$, they can be looked up in numerical tables (for example, H. E. Salzer, “Zeros of the derivative of Bessel functions of fractional order,” Math. Tables Aids Comput., vol. 7, pp. 69-71, 1953) or computed by software such as Matlab. The first few are found to be (to three decimal places):

$$j'_{0,1} = 3.832; \quad j'_{2/3,1} = 1.401; \quad j'_{4/3,1} = 2.258; \quad j'_{2,1} = 3.054$$

The lowest cutoff wavenumber thus occurs for $l = 1, m = 1$, and is about 25% smaller than the value of $k_c a = j'_{11} = 1.841$ of the dominant TE$_{11}$ mode of the full circular waveguide. You might well ask how it is possible for a waveguide whose area is smaller than that of the full circle waveguide and whose cross section is completely contained within that of the larger waveguide, to have a lower cutoff frequency. The answer is that the field is spread out (in a certain sense) over a wider transverse distance, as roughly sketched for the transverse electric below.

For the TE$_{11}$ mode of the full circular waveguide, the field takes the “shortest path” directly across the circle, while for the $270^\circ$ waveguide, it must take a longer route.