We first need to find the line parameters of a perfectly conducting wire over a perfectly conducting ground plane. These are found from the results of section 8.3.2 using image theory. If we place a perfectly conducting plane at $x = 0$ in Fig. 8.8, the fields will remain the same in $x > 0$. Thus, the current and charge on the remaining wire conductor are unchanged, but the voltage and magnetic flux between the wire and the ground plane are only half of what they were for the two-wire line. Thus, the inductance per unit length and the characteristic impedance are half of what they were for the two-wire line, while the capacitance per unit length is doubled (see (8.58)):

$$Z_c = \frac{\zeta}{2\pi} \ln \frac{h + \sqrt{h^2 - a^2}}{a}; \quad l = \frac{\mu}{2\pi} \ln \frac{h + \sqrt{h^2 - a^2}}{a}; \quad c = \frac{2\pi\varepsilon}{\ln \frac{h + \sqrt{h^2 - a^2}}{a}}$$

We may now use the incremental inductance rule to find the line parameters when the earth is finitely conducting. Because the wire conductivity is different than that of the earth, the form of the incremental inductance rule must be modified. If we trace back through the derivation, we find that the rule must have the form

$$\alpha \simeq \frac{1}{2\mu Z_0} \left[ R_{S,\text{wire}} \frac{\delta l}{\delta n} \bigg|_{\text{wire only}} + R_{S,\text{earth}} \frac{\delta l}{\delta n} \bigg|_{\text{earth only}} \right]$$

instead of (9.77), where the subscripts “wire only” and “earth only” indicate that only the indicated conductor surface is to be incremented, and not the other. Pushing into the ground plane by a distance $\delta n$ effects the change $h \rightarrow h + \delta n$, so we have

$$\frac{\delta l}{\delta n} \bigg|_{\text{earth only}} = \frac{\partial l}{\partial h} = \frac{\mu}{2\pi} h + \sqrt{h^2 - a^2} \left( 1 + \frac{h}{\sqrt{h^2 - a^2}} \right) = \frac{\mu}{2\pi \sqrt{h^2 - a^2}}$$

Pushing into the wire by a distance $\delta n$ effects the change $a \rightarrow a - \delta n$, so

$$\frac{\delta l}{\delta n} \bigg|_{\text{wire only}} = -\frac{\partial l}{\partial a} = -\frac{\mu}{2\pi} \left[ \frac{1}{h + \sqrt{h^2 - a^2}} \left( -\frac{a}{\sqrt{h^2 - a^2}} \right) - \frac{1}{a} \right] = \frac{\mu}{2\pi a \sqrt{h^2 - a^2}}$$

In our problem, $\mu_c = \mu = \mu_0$, so from $(*)$,

$$\alpha \simeq \frac{1}{2\zeta Z_0 \sqrt{h^2 - a^2}} \ln \frac{h + \sqrt{h^2 - a^2}}{a} \left[ \frac{h}{a} R_{S,\text{wire}} + R_{S,\text{earth}} \right]$$

where

$$R_{S,\text{wire}} = \sqrt{\frac{\omega \mu_0}{2\sigma_{\text{wire}}}}; \quad R_{S,\text{earth}} = \sqrt{\frac{\omega \mu_0}{2\sigma_{\text{soil}}}} = R_{S,\text{wire}} \sqrt{\frac{\sigma_{\text{wire}}}{\sigma_{\text{soil}}}} \simeq 316.2 R_{S,\text{wire}}$$

Numerical evaluation shows that this function, for fixed $h$, has a minimum when $a \simeq 0.0128 h$ (this must be determined numerically, as there is no closed-form analytical solution). Substituting this value into the expression for $Z_c$ above, we find

$$Z_c \simeq \frac{\zeta}{2\pi} \ln \frac{1 + \sqrt{1 - 0.0128^2}}{0.0128} = 303 \, \Omega$$