Name Solution

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Closed book and closed notes. Put your name on each answer sheet and write on one side only. Remember to show all your work; correct final answers are not enough. Formulas are on the last page.

(24 points)

1. Consider the following equation of motion of an electron in an ac field:

$$m\frac{d^2x(t)}{dt^2} + \frac{m}{\tau}\frac{dx(t)}{dt} = -e\mathcal{E}(t)$$

a. Solve this equation for the case of $\omega << 1/\tau$ (equivalent to $\omega\tau << 1$) to derive the low frequency dielectric function $\epsilon(\omega)$. (Hint: once you have solved for the electron motion, use the polarization density to get you to ϵ. Then use the fact that ω is small to simplify.)

b. What is the phase of the polarization relative to $\mathcal{E}(t)$ in this frequency range? State if the polarization is leading or lagging.

c. What is the phase of the electron current relative to $\mathcal{E}(t)$ in this frequency range? State if the current is leading or lagging.

d. Is the material absorptive in this frequency range? Explain physically.

a. Harmonic solution : $x(t) = x_0 \exp(-i\omega t)$ & $\mathcal{E}(t) = \mathcal{E}_0 \exp(-i\omega t)$

Substitute into equation & divide by $\exp(-i\omega t)$:

$$m(-i\omega)^2 x_0 + \frac{m}{\tau}(-i\omega)x_0 = -e\mathcal{E}_0$$

$$\frac{\omega m x_0}{\tau}(-\omega\tau - i) = e\mathcal{E}_0 \quad (\omega\tau \to 0)$$

$$x_0 = -i\frac{e\mathcal{E}_0\tau}{m\omega}$$

$$\epsilon_r(\omega) = 1 + \chi(\omega) = 1 + \frac{P_0}{\mathcal{E}_0} = 1 - \frac{nex_0}{\mathcal{E}_0} = 1 + \frac{ine^2\tau}{m\omega}$$

$$\cong \frac{ine^2\tau}{m\omega}$$

b. $P_0 = -nex_0 = +i\frac{ne^2\mathcal{E}_0\tau}{m\omega}$ From the "i" we see that P leads $\mathcal{E}$ by $90°$

c. $$I \propto \frac{dP}{dt} = i\frac{ne^2\mathcal{E}_0\tau}{m\omega}\frac{d}{dt}\exp(-i\omega t)$$

$$= (-i\omega)\, i\frac{ne^2\mathcal{E}_0\tau}{m\omega}\exp(-i\omega t) = \frac{ne^2\tau}{m}\mathcal{E}(t)$$

The current is in phase with $\mathcal{E}(t)$

d. Because the current is in phase with the field (and hence the voltage), there is power dissipation. Therefore the material is absorptive in this range.

Name Solution

(14 points)

2. An n-type semiconductor is illuminated with 30 μm infrared light. For what dopant concentrations will the semiconductor be transmissive (non-reflecting)? Assume that all the dopants are activated, i.e., the electron concentration is equal to the dopant concentration, that $\epsilon_r = 7$, and that the effective mass $m^* = 0.2\, m_e$.

Find electron concentration for plasma frequency corresponding to 30 μm.

$$\omega_p^2 = \frac{ne^2}{\epsilon_\infty \epsilon_r m^*} \qquad \text{where } \omega = 2\pi f = \frac{2\pi c}{\lambda}$$

$$n = \frac{\omega_p^2 \epsilon_\infty \epsilon_r m^*}{e^2} = \frac{4\pi^2 c^2 \epsilon_\infty \epsilon_r m^*}{\lambda^2 e^2}$$

$$= \frac{4\pi^2 (3\times 10^8 \frac{m}{s})^2 \, 8.85\times 10^{-12}\frac{F}{m} \times 7 \times 0.2 \times 9.1\times 10^{-31}\,kg}{(30\,\mu m)^2 \times (10^{-6}\frac{m}{\mu m})^2 (1.6\times 10^{-19})^2}$$

$$= 1.74\times 10^{24}\, m^{-3} \times 10^{-6}\,\frac{m^3}{cm^3}$$

$$= 1.7\times 10^{18}\, cm^{-3}$$

For n equal to this value the material is transmissive for $\lambda \le 30\,\mu m$. For n less than this concentration the material will be transmissive at 30 μm.

$$n \le 1.7\times 10^{18}\, cm^{-3}$$

Name Solution

(20 points)

3. In an amorphous semiconductor there is no k-space, and hence all valence band to conduction band transitions are allowed without the need for phonon emission or absorption.
   a. Find an expression which is proportional to the joint density of states as a function of photon energy $\hbar\omega$ for optical transitions in an amorphous semiconductor. This semiconductor has a bandgap E_g, a conduction band density of states, $g_c(E) \propto (E - E_g)^{1/2}$, and a valence band density of states $g_v(E) \propto |E|^{1/2}$, where $E = 0$ at the top of the valence band. Set up and simplify the integral completely, but do not solve it.
   b. Would you expect the absorption coefficient α for amorphous semiconductor to be greater or less than that for crystalline semiconductors having an indirect gap? Why? Assume that $\hbar\omega > E_g$ in both cases.
   c. Would you expect the absorption coefficient α for amorphous semiconductor to be greater or less than that for crystalline semiconductors having a direct gap? Why? Again, assume that $\hbar\omega > E_g$ in both cases.

a. $g_v(E_i) \propto |E_i|^{1/2}$ $\qquad \hbar w = E_f - E_i$

$\Rightarrow E_f = \hbar w + E_i$

$g_c(E_i) \propto (E_f - E_g)^{1/2}$

$\propto (\hbar w + E_i - E_g)^{1/2}$

Note: Initial state is in VB, and is by definition negative

Joint density of states $\propto \int_0^{E_g - \hbar w} g_c(E_i)\, g_v(E_i)\, dE_i$

$\propto \int_0^{E_g - \hbar w} (-E_i)^{1/2} (\hbar w + E_i - E_g)^{1/2}\, dE_i$

$\propto \int_0^{E_g - \hbar w} [-E_i^2 - E_i(\hbar w - E_g)]^{1/2}\, dE_i$

b. Greater. Because there is no k-space for amorphous semiconductors there is no need for phonon participation in the absorption process, and so the transition probability is higher.

c. Greater. In direct gap crystalline semiconductors absorption is dominated by vertical transitions (no change in k). These have a small joint density of states. In amorphous semiconductors all valence-band to conduction-band transitions are vertical (because no k-space) and so the joint density of states is large.

Name Solution

(14 points)

4. Derive the density of states in the x-y plane of a two-dimensional quantum well, g^{2d}, starting with the density of states in the x-y plane per unit k-area:

$$g^{2d}(k_{xy}) = 2\,(1/2\pi)^2$$

You may use the fact that

$$E = \frac{\hbar^2}{2m}\,(k_{xy}^2 + k_z^2\,).$$

$$g^{2d}(E)\,dE = \underbrace{g^{2d}(k_{xy})}\,\underbrace{d^2k_{xy}} \quad \text{(square)}$$

$$= \frac{2}{(2\pi)^2}\; 2\pi\, k_{xy}\, dk_{xy} \quad \text{(annulus)}$$

$$g^{2d}(E) = \frac{k_{xy}}{\pi}\;\frac{1}{dE/dk_{xy}}$$

$$E = \frac{\hbar^2}{2m}\left(k_{xy}^2 + k_z^2\right) = \frac{\hbar^2}{2m}\,k_{xy}^2 + E_z$$

$$\frac{dE}{dk_{xy}} = \frac{\hbar^2 k_{xy}}{m}$$

$$\therefore\; g^{2d}(E) = \frac{k_{xy}}{\pi}\;\frac{1}{\hbar^2 k_{xy}/m} = \underline{\frac{m}{\hbar^2 \pi}}$$

(7 points)

5. What types of electromagnetic waves are supported in a region that is free of external charge? Describe two possibilities.

From $\vec{\nabla}\cdot\vec{D} = 0$ we find

1) If $\epsilon \neq 0$ (the usual case), then $\vec{k} \perp \vec{\mathcal{E}}$, i.e., transverse electric waves

2) If $\epsilon = 0$ (at plasma frequency), $\vec{k} \parallel \vec{\mathcal{E}}$ is allowed, i.e., longitudinal waves.

Name Solution

(7 points)

6. What assumptions are made in deriving Fermi's Golden Rule?

i) Transition probability $P \ll 1$

ii) t (time) is sufficiently small that perturbation is weak, but large enough that energy width of P is small compared to the variation in the density of states

(7 points)

7. Explain qualitatively why the reflectivity of NaCl in the middle of the restrahlen band is observed to decrease from 98% at 100 K to 90% at 300 K.

The peak reflectivity is dominated by γ, the damping constant. It increases as the temperature increases.

(7 points)

8. The energy density of zero-point energy electromagnetic modes in vacuum is proportional to ω^3. Why is that significant? (There are several possible answers.)

This is a Lorentzian distribution. The result is that there is no apparent redshift or blue shift if the source is moving away or towards the observer.

It means that the energy density increases greatly with frequency and there is a huge energy density at high frequencies.

It means that Casimir cavities must have very closely spaced plates to exclude a significant part of the spectrum

FORMULAS

$\nabla \bullet \mathbf{D} = \rho = 0$ (in absence of external charge)

$$\nabla \bullet \mathbf{B} = 0$$

$$\nabla \times \boldsymbol{\varepsilon} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$

$$\nabla \bullet \nabla \times \mathbf{A} = 0$$

$$\nabla \times \nabla \times \boldsymbol{\varepsilon} = \nabla\,(\nabla \bullet \boldsymbol{\varepsilon}) - \nabla^2 \boldsymbol{\varepsilon}$$

$$\mathbf{D} = \epsilon\, \boldsymbol{\varepsilon} = \epsilon_0 \boldsymbol{\varepsilon} + \mathbf{P}$$

$$\mathbf{P} = \chi\, \boldsymbol{\varepsilon}$$

$$P = -\,nex(t)$$

$$\epsilon(\omega) = \epsilon_1(\omega) + i\,\epsilon_2(\omega) = 1 + \chi(\omega)$$

$$T = \sin^2 2\Omega \sin^2 \frac{\pi d \Delta n}{\lambda}$$

$$E_B = -\frac{e^2}{2a_B} = 13.6 \text{ eV}$$

$$E_r = -E_{B,n} \frac{1}{n^2 \epsilon^2} \frac{m_r}{m_e} = -\frac{e^2}{2\epsilon\, a_x} \frac{1}{\eta^2}$$

$$\mathbf{J} = \sigma \boldsymbol{\varepsilon}$$

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0 \exp i(\mathbf{k} \bullet \mathbf{r} - \omega t)$$

$$\omega_p^{\ 2} = \frac{ne^2}{\epsilon_\infty \epsilon_0\, m^*}$$

$$\omega^2 = \frac{\omega_p^{\ 2}}{2}\left[1 + \frac{2c^2 k_x^{\ 2}}{\omega_p^{\ 2}} - \sqrt{1 + \frac{4c^4 k_x^{\ 4}}{\omega_p^{\ 4}}}\right]$$

$\epsilon_0 = 8.85 \times 10^{-12}$ F/m

$e = 1.6 \times 10^{-19}$ C

$c = 3.00 \times 10^{8}$ m/s

$m_e = 9.1 \times 10^{-31}$ kg

$\lambda\,(\mu m) \approx 1.24 / \hbar\omega$ (eV)

$$a_B = \frac{\hbar^2}{m_e e^2}$$

$$a_x = \frac{\epsilon \hbar^2}{m_r e^2}$$