**Photorefractive Effect**

1. Photorefractive effect
2. Volume Holography
3. Kukhtarev band transport equations
4. Two-wave mixing
5. Anisotropic Electrooptic effect
6. Photorefractive Grating Recording and Readout
7. Photorefractive beam fanning
8. Photorefractive Oscillators
9. Photorefractive Four-Wave Mixing
10. Self-Pumped Phase Conjugation

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**Photorefractive Materials**

<table>
<thead>
<tr>
<th>Crystal Type of material class</th>
<th>unique</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fast responding photoconductors</td>
<td>Optical</td>
</tr>
<tr>
<td>BSO, BGO, BTO</td>
<td>paraelectric sellenites</td>
</tr>
</tbody>
</table>

High Efficiency
- BaTiO$_3$, KNbO$_3$ ferroelectric perovskites 4mm huge $r_{42}$
- SBN, BSKNN ferroelectric tungsten bronze 4mm huge $r_{33}$

Slow, semipermanent storage, photogalvonic transport
- LiNbO$_3$:Fe, LiTaO$_3$ Lithium Niobate 3m huge $\beta$

High speed due to high mobility, efficient transport.
- GaAs:Cr, InP:Fe semi-insulating semiconductors 43m large $\mu$
- Sn$_2$P$_2$S$_6$ non-oxide ferroelectric, IR $m$ biaxial

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**Photorefractive Band Transport**

\[ I(x) = I_0(1 + m \cos K_g x) \] Intensity grating
\[ m = \frac{2 \sqrt{I_1 I_2}}{I_1 + I_2} \]
\[ n(x) = n_0(1 + M \cos K_g x) \] electron density
\[ M = \frac{m}{1 + \beta / s I_0} \]

Space Charge
\[ M = .8, .6, .4, .2 \]

Space Charge Field
\[ M = .8, .6, .4, .2 \]

Index grating $\Delta n$ 90° shifted

1. Initial distribution of ionized traps is controlled by oxidation/reduction treatment and is compensated by an equal number of acceptors
2. Incident illumination photoexcites electron from occupied trap ($N_D - N_D^+$) to the mobile conduction band. Trap is now ionized ($N_D^+$)
3. Electron in conduction band moves under the influence of drift and diffusion.
   Electron can be considered to have hopped from one trap to another.
Coupled Mode Equations

Coupled Mode Theory for Volume Diffraction

\[
\begin{align*}
\frac{\partial A_1}{\partial z} &= i\kappa A_2 e^{i\Delta k z} \\
\frac{\partial A_2}{\partial z} &= i\kappa^* A_1 e^{-i\Delta k z}
\end{align*}
\]

At the Bragg angle \( \theta_B = \sin^{-1}\left(\frac{\lambda}{k}\right) \)

I.C. \( A_2(0) = 0 \), so

\[
A_1(x) = A_1(0) \cos |\kappa|z
\]

\[
A_2(x) = -i\kappa^* A_1(0) \sin |\kappa|z
\]

DE \( \eta = 100\% \) at \( \kappa L = \pi \) is limited by other orders to 90-95%

Bragg Mismatched Readout

\[
\begin{align*}
\frac{\partial A_1}{\partial z} &= -i\kappa A_2 e^{i\Delta k z} \\
\frac{\partial A_2}{\partial z} &= -i\kappa^* A_1 e^{-i\Delta k z}
\end{align*}
\]

constant of system (by conservation of energy)

\[
\frac{\partial}{\partial z}(|A_1|^2 + |A_2|^2) = 0
\]

When \( A_2(0) = 0 \)

solution

\[
\begin{align*}
A_1(z) &= e^{i\Delta k z/2} \left[ \cos sz - \frac{i\Delta k}{2s} \sin sz \right] A_1(0) \\
A_2(z) &= e^{-i\Delta k z/2} \left[ -\frac{i\kappa^*}{s} \sin sz \right] A_1(0) e^{-i\Omega t}
\end{align*}
\]

\[
s^2 = \kappa^* \kappa + \left(\frac{\Delta \theta}{\theta_B}\right)^2
\]
Momentum space (⃗k-space and ⃗K-space)

3-D Fourier space

Coupled wave equation with a static dielectric perturbation \( \delta \varepsilon(\vec{r}) \)

\[
\nabla^2 [E_i(\vec{r}) + E_d(\vec{r})] + \frac{\omega^2}{c^2} [\varepsilon_r + \delta \varepsilon(\vec{r})] [E_i(\vec{r}) + E_d(\vec{r})] = 0
\]

Separate into the coupled equations for the incident field and the diffracted field.

\[
\nabla^2 E_i(\vec{r}) + \frac{\omega^2}{c^2} E_i(\vec{r}) = -\frac{\omega^2}{c^2} \delta \varepsilon(\vec{r}) E_d(\vec{r}) \approx 0
\]

\[
\nabla^2 E_d(\vec{r}) + \frac{\omega^2}{c^2} E_d(\vec{r}) = -\frac{\omega^2}{c^2} \delta \varepsilon(\vec{r}) E_i(\vec{r})
\]

Born approximation: incident wave \( E_i \) is much stronger than the diffracted wave \( E_d \).

Represent the diffracted wave as a transverse Fourier expansion of its plane wave components in the unperturbed media, which evolves along the nominal direction of propagation \( z \).
Momentum space derivation continued

Take transverse Fourier transform in $x$ and $y$

$$e^{ik_zd(k_x, k_y)z} \frac{\partial}{\partial z} E_d(k_x, k_y, z) = \frac{i\omega^2}{2c^2k_{zd}(k_x, k_y)} \int \int \delta(\vec{r}) E_i(\vec{r}) e^{-i(k_x x + k_y y)} dx dy$$

Integrate directly to yield the field of the diffracted wave $E_d$ at the exit face, $z = L$

$$E_d(k_x, k_y, L) = \frac{i\omega^2}{2c^2k_{zd}(k_x, k_y)} \int \int \int_{z=0}^{z=L} \mathcal{F}_{xy} \delta(\vec{r}) E_i(\vec{r}) e^{-i(k_x x + k_y y)} dz$$

Reformulated as a 3-D Fourier transform by noting that $\delta(\vec{r})$ vanishes outside the region $z \in \{0, L\}$.

$$E_d(k_x, k_y, L) = \frac{i\omega^2}{2c^2k_{zd}(k_x, k_y)} \int \int \int_{z=0}^{z=L} \mathcal{F}_{xy} \delta(\vec{r}) E_i(\vec{r}) e^{-i(k_x x + k_y y)} dz$$

This is key result. It states that the angular spectrum components of the diffracted field at the output of the media containing the weak dielectric perturbations is given by the 3-D Fourier transform of the product of the incident field and the dielectric perturbation, evaluated on the surface of the allowed propagating modes for the diffracted field.

Angular Selectivity

Recording Bragg Matched Readout

To first order momentum surface is a flat surface tilted by $\pm \theta$ on both input and output beams. Rotating input beam away from exact Bragg matching angle by $\Delta \theta$ will give a $z$ component to the motion of the $k$ vectors given by

$$\sin \theta = \frac{\delta k_z}{s} \quad \text{where} \quad s = \frac{2\pi n}{\lambda} \Delta \theta \quad \implies \quad \delta k_z = \frac{2\pi n}{\lambda} \Delta \theta \sin \theta$$

vector sum of $\vec{k} + \vec{k}_G$ shifts away from the output surface by $\delta k_z$ too giving

$$\Delta k_z = 2\delta k_z = \frac{2\pi n}{\lambda} \Delta \theta \sin \theta$$

The intensity $DE$ is given by power of sinc function evaluated on momentum surface

$$\eta(\Delta \theta) = |\phi|^2 \sin^2 \left( \frac{2\pi n}{\lambda} \Delta \theta L \right) = |\phi|^2 \sin^2 \left( \frac{2\pi n}{\lambda} \sin \theta \Delta \theta L \right)$$

with the first zero of the sinc null at $\theta_0 = \frac{L \omega_{ED}}{2\pi n} \sin \phi$ and $\phi = \frac{\pi n L}{\lambda \cos \theta}$

Grating Uncertainty

Finiteness of volume leads to a distribution of grating vectors due to Fourier uncertainty

Bragg matching

Angular selectivity

Wavelength selectivity

Wavelength selectivity of volume gratings

Recording Wavelength Shifted Readout

Alignment for maximum $DE$ at $\lambda_1$ then read-out at shifted wavelength $\lambda_2$. Produces $k$ offset $p = \sin \theta (|\vec{k}_2| - |\vec{k}_1|)$. Grating vector of length $K_G + 2p$ would be Bragg matched.

Leads to a phase mismatch in $z$

$$\Delta k_z = 2p \tan \theta = 2 \tan \theta \sin \theta (|\vec{k}_2| - |\vec{k}_1|)$$

$$= 2 \sin^2 \theta \left( \frac{2\pi}{\lambda_2} - \frac{2\pi}{\lambda_1} \right)$$
Holography Basics

Writing, interfere object wave with reference wave

\[ I_w(\vec{r}) = |E_o(\vec{r}) + E_r(\vec{r})|^2 = |E_o(\vec{r})|^2 + |E_r(\vec{r})|^2 + E_o^*(\vec{r})E_r(\vec{r}) + E_o(\vec{r})E_r^*(\vec{r}) \]

Record index hologram proportional to intensity

\[ t(\vec{r}) = \Delta n(\vec{r}) = \kappa \int_0^T I_w(\vec{r},t)dt \]

Readout with Reference wave

\[ E_d = \kappa \tau t(\vec{r})E_r = \kappa \tau E_r \left[ |E_o(\vec{r})|^2 + |E_r(\vec{r})|^2 + E_o^*(\vec{r})E_r(\vec{r}) + E_o(\vec{r})E_r^*(\vec{r}) \right] \]

= \kappa \tau \left[ (|E_o(\vec{r})|^2 + |E_r(\vec{r})|^2) E_r + E_o^*E_r + E_oE_r^* \right]

DC order Conjugate Object

Illuminate hologram with phase conjugate reference

Produces a real image

Allows distortion compensating readout

Can illuminate with counterpropagating plane wave

or expanding spherical wave ref can be conjugated as focusing spherical wave

Band Transport Equations: Kuktaev

\[ \frac{\partial N_D^+}{\partial t} = (sI + \beta)(N_D - N_D^+) \]

\[ = \gamma_R N_D^+ q e \]

\[ - \gamma_R N_D^+ \]

\[ = \gamma_R N_D^+ \]

\[ \frac{\partial n_e}{\partial t} - \frac{\partial N_D^+}{\partial t} = \frac{1}{q} \nabla \cdot \vec{J} \]

\[ \vec{J} = q \mu T \nabla n_e + k_B \mu T \nabla n_e \]

\[ \epsilon \nabla \cdot \vec{E} = -q \left( n_e + N_A - N_D^+ \right) \]

- \( N_D^+ \) - ionized donor density
- \( s \) - ionization rate constant
- \( I \) - optical intensity
- \( \beta \) - dark generation rate
- \( N_D \) - total donor site density
- \( \gamma_R \) - recombination rate
- \( n_e \) - total electron density
- \( q \) - electron charge
- \( \vec{J} \) - current density
- \( \mu \) - carrier (electron) mobility
- \( \vec{E} \) - space-charge field
- \( k_B \) - Boltzman’s constant, \( T \) - temp
- \( \epsilon \) - DC dielectric constant
- \( N_A \) - acceptor density

Continuity for immobile donors

\[ \frac{\partial N_D^+}{\partial t} = G = R = (sI + \beta)(N_D - N_D^+) \]

**Generation Rate** is given by the number of donors not ionized (the total) - ionized time the absorprion cross section \( s \) [cm²/W/s] time intensity \( I \) [W/cm²] plus the thermal generation rate \( \beta [1/s] \).

**Recombination Rate** is given by the recombination rate constant \( \gamma_r \) [cm³/s] times the free carrier (electrons or holes) density \( n \) [cm⁻³] time the density of ionized donors \( N_D^+ \) [cm⁻³].
Continuity equation for Electrons

\[ \frac{\partial n}{\partial t} = \frac{\partial N_D^+}{\partial t} - \frac{1}{e} \nabla \cdot \vec{J} \]

Balance between the generation of mobile carriers and the divergence of the current leaving a differential volume.

Current = Drift + Diffusion + Photogalvanic

\[ \vec{J} = e \mu \vec{E} + k_B T \mu \nabla n + \vec{J}_{ph} \]

Total Field = Applied Field + Space Charge Field

\[ \vec{E} = \vec{E}_A + \vec{E}_{sc} \]

Anomalous Photogalvanic Current

\[ J_{ph}^i = \beta_{ijk} \tilde{E}_j \tilde{E}_k \]

\[ \text{I}(\vec{r}, t) = \text{Intensity Profile} \]

s = photo-ionization cross-section
n = optical index
n_o = ordinary
n_e = extraordinary
\( \epsilon \) = DC dielectric tensor
\( \mu \) = mobility tensor
\( \gamma \) = impermeability tensor
\( \Delta \gamma \) = impermeability tensor perturbation
\( \epsilon_o \) = ordinary dielectric tensor
\( \epsilon_e \) = extraordinary dielectric tensor
\( \Delta \epsilon \) = optical dielectric tensor perturbation
E = optical field
p = optical polarization
r = electrooptic tensor
β = photogalvanic tensor

Optical Equations

Electro-optic index modulation

\[ \Delta n = -\frac{n^2}{2} r_{eff} E \]

Effective electro-optic coefficient

\[ r_{eff} = \hat{p}_i \epsilon_{ikl} r_{klm} \hat{K} G_m^r \epsilon_{lj} \hat{p}_j \]

More formally

\[ \Delta \eta_{kl} = r_{klm} E_m \]

Perturbation of the dielectric tensor

\[ \Delta \epsilon^r_{ij} = \epsilon^r_{ikl} r_{klm} E_m \epsilon^r_{mj} \]

Wave Eqn

\[ \nabla^2 \tilde{E} - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} (\epsilon \tilde{E}) = 0 \]

\[ \tilde{E}_m = \sum_q \tilde{E}_m(q) e^{i(\omega_q t - \phi_q(\vec{r}))} + cc \]

\[ I(\vec{r}, t) = \sum_m |\tilde{E}_m|^2 \frac{\epsilon_0 c n}{2} \]

Gauss’ Law

\[ \nabla \cdot D = \rho \]

where

\[ N_D \]

total trap density

\[ N_D^+(\vec{r}, t) = \text{ionized trap density} \]

\[ N_D - N_D^+(\vec{r}, t) = \text{occupied trap density} \]

\[ N_A = N_A^+(I = 0) \text{ compensating acceptor density} \]

\[ n(\vec{r}, t) \]

electron density

\[ p(\vec{r}, t) \]

hole density

\[ \vec{E}_A \]

applied E field

\[ \vec{E}_{sc}(\vec{r}, t) \]

space charge field

\[ \vec{J}(\vec{r}, t) \]

current density

\[ D = k_B T \mu / e \]

diffusion constant

\[ \beta \]

thermal ionization rate

\[ \gamma_r \]

recombination rate

Definitions
Solution of the Kuktarev eqns

Uniform Illumination

\( I(x) = I_0 \Rightarrow \text{by symmetry solns constant for } N_D^+ \), \( n, E \). From Gauss’ law

\[ \frac{\nabla \cdot \epsilon E}{\epsilon} = 0 = (N_D^+ - N_A - n) = 0 \] (local neutrality)

In continuity all \( \nabla \cdot = 0 \)

\[ \frac{\partial n}{\partial t} = \frac{\partial N_D^+}{\partial t} + 0 = (N_D - N_A - n)(sI + \beta) - \gamma_r(N_a + n)n \]

Assume \( n \ll N_A \ll N_D - N_A \)

\[ \frac{\partial n}{\partial t} = (N_D - N_A)(sI + \beta) - \frac{n}{\tau_r} \]

\( \tau_r^{-1} = N_A\gamma_r \)

\( n_d = \beta(N_D - N_A)\tau_r \)

\( n = n_d + n \approx n_d \left( 1 - e^{-t/\tau_r} \right) \)

\[ n = (n_0 - n_d) \left( 1 - e^{-t/\tau_r} \right) \]

Boundary Conditions

\[ \frac{V}{L} = \int_0^L E \, dx = \frac{J}{q\mu_0} \int_0^L \frac{1}{1 + m \cos Kx} \, dx + \frac{DK}{\mu} \int_0^L \frac{m \sin Kx}{1 + m \cos Kx} \, dx \]

\[ = \frac{J}{q\mu_0} \frac{1}{\sqrt{1 - m^2}} \] very large no. of fringes

\[ J = \sqrt{1 - m^2} q\mu_0 E_a, \quad J = \sigma E_a \]

\( \sigma_0 = \text{photoconductivity} \)

\[ \frac{\partial n}{\partial t} = \frac{\partial N_D^+}{\partial t} - \nabla \cdot (\mu n E + D \nabla n) \]

**steady state**

\[ n = \frac{(N_D - N_A)(sI + \beta)}{\gamma_r N_A} = n_0(1 + m \cos \vec{K}_G \cdot \vec{x}) \]

\[ m = \frac{M}{1 + \beta/s_0} \]

\[ \frac{\partial n}{\partial t} = \frac{\partial N_D^+}{\partial t} = \frac{\partial}{\partial x} \left( D \frac{\partial}{\partial x} (n + m \mu E) \right) = 0 \]

Integrate wrt \( x \) gives \( f \) as constant of integration \( E = \frac{2 - \frac{D q\mu}{\mu_0}}{m} \)

\[ n = n_0(1 + m \cos K_G \cdot x) \Rightarrow \frac{\partial n}{\partial x} = -n_0 m K_G \sin K_G x \]

\[ n_0 = \frac{(N_D - N_A)(sI_0 + \beta)}{\gamma_r N_A} \]

**Drift Diffusion**

\[ E_{sc} = \frac{J}{q\mu_0} \frac{1}{1 + m \cos Kx} + \frac{DK}{\mu} \frac{m \sin Kx}{1 + m \cos Kx} = E_A \sqrt{1 - m^2} + E_D \frac{m \sin Kx}{1 + m \cos Kx} \]

**Fourier Decompose**

\[ E = \sum_l E_l e^{iK_G x} \]

\[ E_0 = E_A, \quad E_{\pm l} = E_l \] since space charge field is real

\[ E_l = (E_A + i E_D)(-1)^l \left[ 1 - \frac{1 - m^2}{m} \right] \]

\( l = \pm 1 \) are only Bragg matched harmonics for volume hologram

\[ E_G = E_A e^{iK_G x} + cc = -2 \left[ 1 - \sqrt{1 - m^2} \right] \frac{E_A^2 + E_D^2}{m} \frac{1}{2} \cos(K_G x + \phi) \]

\( \phi = \tan^{-1} \frac{E_D}{E_A} \)

\( \phi \to 90^\circ \) no applied field \( E_A \Rightarrow 2WM \text{ gain} \)

\( \phi \to 0 \) large \( E_A \)

\( E_G \propto m \) for \( m < 1 \)
Variation of the PR index grating phase with applied field

Kelvin Wagner, University of Colorado Nonlinear Optics 2012

Two Wave Mixing

Input fields coupled by Photorefractive nonlinearity

\[ E(\vec{r}, t) = \hat{e}_1 A_1(z) e^{-i(\vec{k}_1 \cdot \vec{r} + \omega_1 t)} + \hat{e}_2 A_2(z) e^{-i(\vec{k}_2 \cdot \vec{r} + \omega_2 t)} \]

Two angled beams interfere

\[ I(\vec{r}, t) = E^* \cdot E = |A_1(z)|^2 + |A_2(z)|^2 + \hat{e}_2 \cdot \hat{e}_2 A_1(z) A_2^*(z) e^{-i(\vec{k}_1 - \vec{k}_2) \cdot \vec{r} + (\omega_1 - \omega_2) t} + cc \]

\[ = I_0(z) \left( 1 + \frac{\hat{e}_2 \cdot A_1(z) A_2^*(z)}{I_0(z)} e^{-i(\vec{k}_1 - \vec{k}_2) \cdot \vec{r} + (\omega_1 - \omega_2) t} + cc \right) \]

Index varies in response to the spatial variations of the intensity grating. Assume \( \Omega = \omega_1 - \omega_2 = 0 \).

\[ n(\vec{r}) = n_0 + n_1 e^{-i\phi} \frac{A_1(z) A_2^*(z)}{I_0(z)} e^{-i(\vec{k}_1 - \vec{k}_2) \cdot \vec{r}} \]

Where

\[ n_1 e^{-i\phi} = r_{\text{eff}} \frac{\kappa^2}{|E_0|^2} E_{\text{ac}} = -i r_{\text{eff}} \frac{\kappa^2}{|E_0|^2} E_0 (E_0 + E_D + E_q) \]

\[ E_0 \] Applied field (including photogalvanic

\[ E_q \] Space charge limiting field

\[ E_D \] diffusion field

\[ \phi \] phase shift

Coupled Mode Equations

\[ \nabla^2 \tilde{E} + \frac{1}{c^2} \frac{d^2 (\varepsilon \cdot \tilde{E})}{dt^2} = 0 \]

monochromatic fields gives

\[ \nabla^2 \tilde{E} + k_0^2 \varepsilon^r \tilde{E} = 0 \]

where \( \varepsilon^r = n_0^2 + i \frac{n_0 n_{\text{eff}}}{k_0} + \Delta \varepsilon^r \)

Plug in assumed E fields, employ SVEA, and separate into coupled eqns

\[ \cos \theta_1 \frac{d A_1}{dz} = -\frac{\alpha}{2} A_1 - i \frac{\pi n_1}{\lambda} e^{-i\phi} A_1(z) A_2^*(z) \frac{I_0(z)}{A_2(z)} \]

\[ \cos \theta_2 \frac{d A_2}{dz} = -\frac{\alpha}{2} A_2 - i \frac{\pi n_1}{\lambda} e^{-i\phi} A_1(z) A_2^*(z) \frac{I_0(z)}{A_1(z)} \]

define a coupling constant \( \gamma = i \frac{\pi n_1}{\lambda} e^{-i\phi} \)

\[ I_0(z) = |A_1(z)|^2 + |A_2(z)|^2 \] constant if \( \alpha = 0 \)
Separate into intensity and phase equations

\[ \cos \theta_1 \frac{dI_1}{dz} = -\alpha I_1 - \frac{2\pi}{\lambda} n_1 \sin \phi \frac{I_1 I_2}{I_0} = -\alpha I_1 + \Gamma \frac{I_1 I_2}{I_1 + I_2} \]
\[ \cos \theta_2 \frac{dI_2}{dz} = -\alpha I_2 - \frac{2\pi}{\lambda} n_1 \sin \phi \frac{I_1 I_2}{I_0} = -\alpha I_2 - \Gamma \frac{I_1 I_2}{I_1 + I_2} \]
\[ \cos \theta_1 \frac{d\phi_1}{dz} = \frac{\pi n_1}{\lambda} \cos \phi \frac{I_2}{I_0} \]
\[ \cos \theta_2 \frac{d\phi_2}{dz} = \frac{\pi n_1}{\lambda} \cos \phi \frac{I_1}{I_0} \]

For a local nonlinearity or PR with large \( E_A \)
\( \phi = 0 \) no energy exchange but fringe curvature.
For a diffusion dominated photorefractive, no fringe curvature.
\( \phi = \frac{\pi}{2} \) PR 90° phase shift leads to energy exchange

Symmetric transmission geometry

\[ I_1 + I_2 = [I_1(0) + I_2(0)] e^{-\alpha r} \] decouples equations
\[ r = \frac{\cos \theta}{c} \]
\[ \Gamma’ = I e^{\alpha r} \] new variables

\[ \text{SOhn} \]
\[ I_1(r) = I_1(0) e^{-\alpha r} \frac{I_1(0) + I_2(0)}{I_1(0) + I_2(0)} = I_1(0) e^{-\alpha r} \frac{I_1(0) + I_2(0)}{I_1(0) e^{-\alpha r} + I_2(0)} \]
\[ I_2(r) = I_2(0) e^{-\alpha r} \frac{I_1(0) + I_2(0)}{I_1(0) + I_2(0)} = I_2(0) e^{-\alpha r} \frac{I_1(0) + I_2(0)}{I_1(0) e^{-\alpha r} + I_2(0)} \]
\[ \Gamma = \frac{2\pi n}{\lambda} \sin \phi \] Intensity coupling constant

Small signal limit
\[ I_2(0) \ll I_1(0) e^{-\alpha r} \] no depletion of \( I_1 \)
\[ I_2(L) = I_2(0) e^{\Gamma - \alpha L} \] gain >4000 achieved

Nonlinear grating evolution and optical propagation eqns

\[ \left( \tau \frac{\partial}{\partial t} + 1 \right) \hat{E}^{SC} = \frac{\partial^2 \hat{E}^{SC}}{\partial x^2} - \frac{\partial I}{\partial x} (I + I_d) \]

- Slow variation of space-charge field: \( \psi \approx 1 \)
- What we can’t deal with: \( \tau_2 \to 0 \)
- Large donor density: \( N_D \gg N_A \)

\[ 2ik \frac{\partial A}{\partial z} + \frac{\partial^2 A}{\partial x^2} + 4k \gamma_0 \hat{E}^{SC} A = 0 \]

- \( E(x, z, t) = A(x, z) e^{i(kz - \omega t)} \)
- Paraxial approximation
- Coupling constant: \( \gamma_0 = 2\omega n^2 r_{\text{eff}} / c \)
- \( r_{\text{eff}} \): effective electro-optic coefficient

Two Wave mixing dynamics simulations

Two-Wave Mixing, 5°

\[ 2k \] weak signal
\[ 4k \] strong pump

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Momentum space description of Volume Hologram Diffraction

Angular Dependence

Gain in photorefractive 2WM (simple approximation when no saturation)

\[ G = \frac{(1 + r)e^{\gamma L}}{1 + re^{\gamma L}} \approx e^{\gamma L} \quad \text{for} \quad r \ll e^{\gamma L} \]

\[ r = \frac{I_1(0)}{I_2(0)} \quad \text{Beam ratio} \]

\[ \gamma = \frac{\pi E_\text{r}}{n\lambda \cos \theta} \quad \text{coupling constant} \]

\[ E = k_B T \frac{k_g}{q} \left( 1 + \frac{K_g}{K_0} \right)^2 (\hat{p}_1 \cdot \hat{p}_2) \]

\[ K_g = |K_g| = |\vec{k}_1 - \vec{k}_2| \]

Magnitude of \( K_g \) which runs out of traps

\[ K_0 = \sqrt{\frac{k_B q}{e}} \]

\[ \epsilon = K_g \cdot e \cdot K_g \]
### Electro-optic effect

\[ \eta = \varepsilon^{-1} = \eta_{0} + r \vec{E} \]

In terms of subscript notation

\[ \Delta \left( \frac{1}{\varepsilon} \right)_{ij} = \Delta \eta_{ij} = r_{ijk}E_{k} \]

Reduced subscript

\[ I = 1, 2, 3, 4, 5, 6 \]

\[ ij = 11, 22, 33, 23, 13, 12 \]

3x3x3 tensor becomes 6x3 matrix

\[
\begin{bmatrix}
11 & 12 & \leftarrow & 13 \\
22 & 23 & \downarrow & 33 \\
\end{bmatrix}
\]

### Optimum grating angle for Electro-optic effect in Photorefractives

\[ \varepsilon^{2} r_{\text{eff}} = \frac{1}{2} \left[ n_{1}^{4} r_{13} (\cos 2\theta - \cos 2\beta) + n_{1}^{4} r_{33} (\cos 2\theta + \cos 2\beta) + n_{2}^{4} n_{12}^{2} \sin^{2} \beta \cos \beta \right] \]

Find maximum \( r_{\text{eff}} \):

\[ \frac{dr_{\text{eff}}}{d\beta} = 0 \quad \frac{dr_{\text{eff}}}{d\theta} = 0 \]

**BaTiO\(_{3}\)**

- \( r_{13} = 24\text{pm/V} \)
- \( r_{33} = 80\text{pm/V} \)
- \( r_{12} = 1640\text{pm/V} \)

**SBN**

- \( r_{13} = 67\text{pm/V} \)
- \( r_{33} = 1340\text{pm/V} \)
- \( r_{12} = 42\text{pm/V} \)

### Optimum diffraction efficiency

Maximize space charge field too

\[ \varepsilon^{2} r_{\text{eff}} = \frac{1}{2} \left[ n_{1}^{4} r_{13} (\cos 2\theta - \cos 2\beta) + n_{1}^{4} r_{33} (\cos 2\theta + \cos 2\beta) + n_{2}^{4} n_{12}^{2} \sin^{2} \beta \cos \beta \right] \]

Find maximum \( r_{\text{eff}} \):

\[ \frac{dr_{\text{eff}}}{d\beta} = 2 \sin \beta \cos^{2} \beta - \sin^{3} \beta = 0 \]

\[ \tan^{2} \beta = 2 \quad \beta_{m} = \tan^{-1} \sqrt{2} = 54.7^\circ \]

**BaTiO\(_{3}\)**

- \( \varepsilon_{1}^{1} = 3600 \)
- \( \varepsilon_{3}^{1} = 135 \)

**SBN**

- \( \varepsilon_{1}^{3} = 500 \)
- \( \varepsilon_{3}^{3} = 3400 \)
Maximize diffraction efficiency as $r_{\text{eff}} E_{\text{sc}}$

BaTiO$_3$ $r_{42}$ dominates. In limit $K_g > K_0$, we can approximate

$$r_{\text{eff}} E_{\text{sc}} = \frac{1}{2} n_e^2 n_{\text{e}}^2 r_{42} \sin^2 \beta \cos \beta \frac{k_B T}{q} \frac{N q^2}{K_g \epsilon_0 k_B T} \epsilon_1 \sin^2 \beta + \epsilon_3 \cos^2 \beta$$

$$= \frac{1}{2} n_e^2 n_{\text{e}}^2 r_{42} \frac{N q}{K_g \epsilon_0} \sin^2 \beta \cos \beta \cos 2\theta$$

SBN, $r_{33}$ dominates

$$r_{\text{eff}} E_{\text{sc}} = \frac{1}{2} n_e^4 n_{\text{e}}^4 r_{33} \cos \beta \frac{N q}{\epsilon_0} \frac{1}{n(\alpha_1) + n(\alpha_2)} k_0 \sin \theta \cos 2\theta$$

Then solve for

$$\frac{d(r_{\text{eff}} E_{\text{sc}})}{d\beta} = 0$$

$$\frac{d(r_{\text{eff}} E_{\text{sc}})}{d\theta} = 0$$

K-space construction for finding angles for Bragg-matching red probe beam

Bragg matching a red probe beam

- Photorefractive gating recorded by green beams.
  - deep in Bragg regime
  - narrow angular selectivity
- Red Bragg match requires precise angular alignment and position overlap
  - adjust angle without coupling to position and vice versa

Photorefractive Beam fanning

- Provides a visualization of direction dependent 2WM gain
- A function of angle of incidence, $\alpha_1$, and diffraction angle.
- Noise source that limits PR 2WM image amplifiers
- Provides self-starting mechanism for self-pumped oscillators
- Center of fanning lobe achieves highest 2WM mixing gain
- Fanning and 2WM amplification in competition
  Amplified beam depletes pump AND fanning
Fanning Simulation


Indirect EO effect and roto-optic contribution

G. Montemezzani, Optical wave manipulation and signal processing in anisotropic photorefractive materials, ETH, Zurich.

Field induces strain through piezoelectric $d_{klm}^E$, adds an EO contribution through photoelastic $p_{ijkl}^E$. Where $r_{ijkl}^E$ is clamped EO

$$\Delta \eta_{ij} = r_{ijkl}^E E_k + p_{ijkl}^E S_{kl}$$

$$S_{kl} = d_{klm}^E E_m$$

$$r_{ijkl}^T = r_{ijkl}^E + p_{ijkl}^E d_{klm}^E$$

Additional contribution asymmetric in $kl$ due to local rotations gives roto-optic

$$\Delta \eta_{ij} = r_{ijkl} E_k + p_{ijkl}^E \frac{\partial u_k}{\partial x_j}$$

Which for static PR grating $\vec{E} = E_{cK}$ New EO tensor

$$\eta_{ij} = E_{cK} \left[ r_{ijkl}^E + p_{ijkl}^E A_{kl}^{-1} B_l \right] \vec{K}_k$$

Corrected Theory of Angular Dependence of Photorefractive Gain and Fanning

G. Montemezzani, Optical wave manipulation and signal processing in anisotropic photorefractive materials, ETH, Zurich.

Stress-strain relation in piezoelectric crystal mediated by Stiffness $C_{ijkl}$ and piezoelectric stress $e_{ijkl}$, while dielectric is modified by piezoelectric

$$T_{ij} = C_{ijkl}^E S_{kl} - e_{ijkl} E_k$$

$$D_i = e_{ijk} S_{jk} + e_{ijkl}^S E_j$$

Static periodic particle displacement grating $\Omega = 0$ polarized $U_0 \vec{u} = \vec{u}$ induces strain

$$\vec{u}(r,t) = U_0 \vec{u} \cos(\vec{K} \cdot \vec{r} - \Omega t - \pi/2)$$

$$S_{ij} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right]$$

Elastodynamic eqn balances forces

$$F_i = \rho \frac{\partial^2 u_i}{\partial t^2} = 0 = \partial T_{ij} = C_{ijkl}^E \frac{\partial u_k}{\partial x_j} \frac{\partial E_k}{\partial x_j}$$

$$\partial S_{ij} \frac{A_{ik}}{C_{ijkl}^E K_j K_l} u_k = \frac{E}{K} e_{ijkl}^S K_k K_j K_j$$

solution for the photorefractive grating induced particle displacement field amplitude

$$u_k = A_{ik}^{-1} B_i E/k$$

Modifies the static dielectric constant that depends on grating vector orientation

$$\epsilon_{eff} = \frac{\rho}{\epsilon_e K E} = \vec{K}_k \vec{K}_j \left[ \epsilon_{ij}^S + \frac{1}{\epsilon_e} e_{ijkl} A_{kl}^{-1} B_l \right]$$

new dielectric tensor

$$\epsilon_{ij}^K \neq \epsilon_{ij}^T = \epsilon_{ij}^S + \epsilon_{klm} d_{klm}$$

Photorefractive Resonators

Kelvin Wagner, University of Colorado Nonlinear Optics 2012

![Photorefractive Resonators](image-url)
Photorefractive Resonators

- Enormous gain in PR 2WM makes PR resonators much easier to operate than laser resonators
- Many topologies are possible
- Ring resonators are nice example of nonlinear dynamical system
- Cavity resonance accomplished by grating phase shift
  - not necessary to control cavity length
  - not necessary to stabilize laser
- Continuously shifting grating leads to Doppler shifted mode (Hz)
- Can study stability and modes by opening and closing iris
- Interesting dynamical effects due to sluggish PR response
- Resonators can appear to daydream and drift with dancing modes
- Novelty filters, reflexive coupling, and other advanced applications

4WM in photorefractive Crystals

\[
\frac{\partial A_1}{\partial z} = -\frac{\gamma}{I_0} (A_1 A_4^* + A_2 A_3^*) A_1 - \alpha A_1
\]
\[
\frac{\partial A_2}{\partial z} = -\frac{\gamma}{I_0} (A_1 A_4^* + A_2 A_3^*) A_2 + \alpha A_2
\]
\[
\frac{\partial A_3}{\partial z} = \frac{\gamma}{I_0} (A_1 A_4^* + A_2 A_3^*) A_3 + \alpha A_3
\]
\[
\frac{\partial A_4}{\partial z} = \frac{\gamma}{I_0} (A_1 A_4^* + A_2 A_3^*) A_4 - \alpha A_4
\]

neglect absorption \( \alpha = 0 \)
solve for \( A_{34} = A_3/A_4 \)

\[
A_{12} = A_1/A_2
\]

Constant to decouple eqns given by BC

\[
A_1 A_2 + A_3 A_4 = \frac{c}{2}
\]
\[
A_1 A_3 - A_2 A_4 = \frac{d}{2}
\]
\[
A_1 A_3^* + A_4 A_4^* = I_1 + I_4 = f_+
\]
\[
A_2 A_2^* + A_3 A_3^* = I_2 + I_3 = f_-
\]
\[
I_1 + I_4 - I_2 - I_3 = f_+ - f_- = f
\]

Undepleted Pump, Transmission Only, SVEA

\[
\frac{\partial A_1}{\partial z} = -\frac{\alpha}{2} A_1
\]
\[
\frac{\partial A_2}{\partial z} = \frac{\alpha}{2} A_2
\]
\[
\frac{\partial A_3}{\partial z} = \frac{\alpha}{2} A_3 + \frac{\gamma}{I_0} (A_1 A_4^* + A_2 A_3^*) A_2
\]
\[
\frac{\partial A_4}{\partial z} = \frac{\alpha}{2} A_4 + \frac{\gamma}{I_0} (A_1 A_4^* + A_2 A_3^*) A_4^*
\]

Use constant to uncouple remaining eqns

\[
\frac{\partial}{\partial z} (A_1^* A_3 - A_2 A_4^*) = 0
\]
\[
A_1^* A_3 - A_2 A_4^* = c
\]

so

\[
\frac{\partial A_3}{\partial z} = \left( \gamma + \frac{\alpha}{2} \right) A_3 - \frac{c A_1^*}{I_0} \Rightarrow A_3 = c \gamma A_1^* \int_0^L \frac{e^{(\gamma + \alpha/2)(z-z')}}{I_0(z')} dz'
\]
**Self-Pumped Phase Conjugation**

Internally generated counterpropagating beams (mutual Phase conjugate)

**Linear** (Yariv, White, Cronin-Golomb)

**Cat** (Feinberg and Hellworth)

Entire resonator in single crystal

Scattering seeds resonator, which - no adjustment. builds up to a strong oscillating mode uses corner of crystal as retroreflector. by stealing power via 2WM, which Only retro in horizontal plane, then act as the counterpropagating reference beams for 4WM based conjugation.
Self-Pumped Phase Conjugation Images

FIG. 6. The input image of (a) a resolution chart and (b)–(d) the reconstructed images produced by a BaTiO₃ TIR mirror. (b) The crystal is in the image plane, the intensity distribution shifts to the right, in the direction of focusing. (c) The same as (b) but with the c axis flipped 180°; the shift is in the opposite direction. (d) The crystal is placed 40 mm before the image plane of the chart.


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Mutual Phase Conjugation

Mutual-Pumped Phase Conjugation Simulation

FIG. 5. Structure of the fields corresponding to mutual phase conjugation and bending of two light beams coupled via a reflecting crystal face (the bird wing geometry).


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