Paraxial Ray Tracing

- y-u transfer and refraction equations
  - y-u ray tracing examples
- Matrix approach to ray tracing
- System and conjugate matrices
Paraxial Ray Tracing
Derivation of refraction and transfer equations

Want to know what happens to rays as they propagate in air and interact with lens and mirrors etc.

To follow ray you have to have $y$ (ray height) and $u$ ray slope.

If propagating need to have distance $d$

If power element, need to have power of element $\phi$

$y & d$ in meters  $U, u'$ in radians  $\phi$ in diopters
Paraxial Ray Tracing
Derivation of refraction and transfer equations

Paraxial tangents

\[-t_k = \frac{y_k}{u_k}, \quad t'_k = \frac{y_k}{-u'_k}\]

Substitute into thin-lens equation

Refraction equation

\[-\frac{u'_k}{y_k} = \phi_k - \frac{u_k}{y_k}\]

\[u'_k = u_k - y_k \phi_k\]

Transfer equation

\[y_{k+1} = y_k + u'_k d'_k\]
## Simple Example

![Diagram of optical system with distances labeled]

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1/F</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>φ</td>
<td>0</td>
<td>1/F</td>
<td>0</td>
</tr>
<tr>
<td>d</td>
<td>s_o</td>
<td>0</td>
<td>s_i</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>( h_o )</th>
<th>( h_o )</th>
<th>( h_o - h_o s_i / F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y(h_o) )</td>
<td>( h_o )</td>
<td>( h_o )</td>
<td>( h_o - h_o s_i / F )</td>
</tr>
<tr>
<td>( u(0) )</td>
<td>0</td>
<td>-( h_o / F )</td>
<td>-( h_o / F )</td>
</tr>
</tbody>
</table>

\[
y_{k+1} = y_k + u'_k d'_k \\
u'_k = u_k - y_k \phi_k \\
y_{k+1} = y_k + u'_k d'_k
\]
Question to be answered: what is the front working distance of a 20x, f=8mm objective when used with a 160 mm tube, as shown?

<table>
<thead>
<tr>
<th>Surface k</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>System</td>
<td>( \phi_k = 1/f )</td>
<td>0</td>
<td>1/8</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>( d'_k )</td>
<td>8.4</td>
<td>168</td>
<td>2</td>
</tr>
<tr>
<td>Axial ray</td>
<td>( y_k )</td>
<td>0</td>
<td>84</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>( u'_k )</td>
<td>10</td>
<td>-1/2</td>
<td>-1/2</td>
</tr>
</tbody>
</table>

Transfer: \( 84 = 0 + 10 d'_0 \).
Dealing with different indices
thin lens

Dealing with different indices of refraction:

\[ n \sin \theta = n' \sin \theta' \]

Snell’s Law

\[ nu \approx n'u' \]

Paraxial approximation

\[ \hat{u} \equiv nu \]

Reduced angle variable

\[ \frac{n}{t} + \frac{n'}{t'} = \frac{1}{f} \]

Gaussian thin lens equation

\[ \hat{t} \equiv \frac{t}{n} \]

Reduced distance variables

We can now write equations involving angle and distance but ignoring changes in index. Whenever we deal with problems with several difference indices, we simply make the above substitutions.
Example with different n’s

\[ f = 10 \text{mm}, f' = 16 \]

\[ h_0, n_1 = 1, n_2 = 1.6 \]

\[ s_o = 20 \text{mm}, s_i = 32 \text{mm} \]

\[
\begin{array}{c|c|c|c}
\varphi & 0 & 1 & 2 \\
\hline
\phi & 0 & 1/10 & 0 \\
\hline
d & S_o/n_1 = 20 & 0 & S_i/n_2 \\
\hline
y(0) & 0 & 20 & 20 - s_i/1.6 \\
\bar{u} = n_1u(1) & 1 & 1 - 20/10 = -1 & -1.6 \\
\end{array}
\]

Check \[ zz' = -nn'F^2 \] (Newton’s form) => \[ z' = -1 \times 1.6(10)^2 / -10 = 16 \Rightarrow s_i = 32 \text{mm} \]
Thick lenses with different indices

Snell’s Law

(1) \( n_1 u_1 = n_2 u_2 \)

Incident on a surface with curvature of \( C \) \((1/R)\) with index \( n_2 \) results in (Smith page 40)

(2) \( n_2 u_2 = n_1 u_1 - y_1 (n_2 - n_1) C \) – refraction equation just for one surface

Propagation inside of material index \( n_k \) with thickness of \( t \) gives

(3) \( y_{k+1} = y_k + t \frac{n_k u_k}{n_k} \) - normal transfer equation
y-nu Tracing
with different indices

Where will the image be located?

First write out the parameters given with the correct sign convention

\[
\begin{align*}
h &= 20\text{mm} \\
L_1 &= -300\text{mm} \\
R_1 &= +50\text{mm} \\
R_2 &= -50\text{mm} \\
R_3 &= \text{plano} \\
N_1 &= 1.0 \\
N_2 &= 1.5 \\
N_3 &= 1.6 \\
N_4 &= 1.0 \\
t_1 &= 10\text{mm} \\
t_2 &= 2\text{mm} \\
C_1 &= 0.02 \\
C_2 &= -0.02 \\
C &= 0 \\
n_1 &= 1.0 \\
n_2 &= 1.5 \\
n_3 &= 1.6 \\
n_4 &= 1.0
\end{align*}
\]
y-nu Tracing
with different indices

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
n & 1 & 1.5 & 1.5 & 1.6 & 1.6 & 1 & 1 \\
t & 300 & 10 & 2 & l' \\
C & +0.02 & -0.02 & 0 \\
\end{array}
\]

Height \( y \) 0 10 10 9.55 9.55 9.49 9.49 0
Slope \( n_\nu \) 0.333 0.333 -0.0666 -0.066 -0.47 -0.47 -0.475 -0.475
Equation used 3 2 3 2 3 2

\[ l' = -9.49 / -0.475 = 199.68 \text{mm} \]

*Arbitrary just start at O and want to find when it crosses the axis at O’
### y-u for Cassegrain mirror system

**Tabular method**

#### Where is the focal point?

<table>
<thead>
<tr>
<th>Radius</th>
<th>Distance index</th>
<th>Ray Height</th>
<th>Slope index (nu)</th>
<th>z = 0.2/-0.002</th>
</tr>
</thead>
<tbody>
<tr>
<td>-200</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>100mm</td>
</tr>
<tr>
<td>-80</td>
<td>-1</td>
<td>-1/100</td>
<td>-0.002</td>
<td>20mm to right of PM</td>
</tr>
</tbody>
</table>

**Remember for mirror**

\[ \varphi = 2n/R \]

\[ y_{k+1} = y_k + u_k' d_k' \]

\[ u_k' = u_k - y_k \varphi_k \]

\[ 0.2 = 1 + (-1)(-0.01)(-80) \]

\[ -0.002 = -0.01 - (0.2)(0.04) \]
Example: The Telescope

Keplerian

Shown in the *afocal* geometry (d=f1+f2). Relaxed eye focuses at ~1m, thus telescope are usually not afocal. Analysis simpler, however.

\[ h_1 - h_1(f_1+f_2)/f_1 = -h_2 \]

Independent of distance
Before and after lenses

\[ M = \frac{h_2}{h_1} = -\frac{f_2}{f_1} \]

\[ y_{k+1} = y_k + u'_k d'_k \]

\[ u'_k = u_k - y_k \dot{\phi}_k \]
Example: The Telescope

Keplerian

Shown in the *afocal* geometry (d=f₁+f₂). Relaxed eye focuses at ~1m, thus telescope are usually not afocal. Analysis simpler, however.

\[ M_\theta \equiv \frac{\beta}{\alpha} \quad \text{Definition of angular magnification} \]

\[ = -\frac{h/f_2}{h/f_1} = -\frac{f_1}{f_2} \quad \text{Via similar triangles} \]

\[ = \frac{1}{M} \quad \text{This is both important and fundamental.} \]
ABCD Matrices
Matrix formulation of ray tracing

\[
\begin{bmatrix}
y_k \\
u'_k
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
-\phi_k & 1
\end{bmatrix} \begin{bmatrix}
y_k \\
u_k
\end{bmatrix} \equiv R_k \begin{bmatrix}
y_k \\
u_k
\end{bmatrix}
\]

Refraction equation
\[u'_k = u_k - y_k \phi_k\]

\[
\begin{bmatrix}
y_{k+1} \\
u_{k+1}
\end{bmatrix} = \begin{bmatrix}
1 & d'_k \\
0 & 1
\end{bmatrix} \begin{bmatrix}
y_k \\
u'_k
\end{bmatrix} \equiv T_k \begin{bmatrix}
y_k \\
u'_k
\end{bmatrix}
\]

Transfer equation
\[y_{k+1} = y_k + u'_k d'_k\]
ABCD Matrices
Matrix formulation of ray tracing

$$\begin{bmatrix} y_K \\ u'_K \end{bmatrix} = R_K T_{K-1} R_{K-1} \ldots T_1 R_1 \begin{bmatrix} y_1 \\ u_1 \end{bmatrix} \equiv M \begin{bmatrix} y_1 \\ u_1 \end{bmatrix}$$

System matrix

$$\begin{bmatrix} y_{K+1} \\ u_{K+1} \end{bmatrix} = T_1 R_1 T_0 \begin{bmatrix} y_1 \\ u'_0 \end{bmatrix} \equiv N \begin{bmatrix} y_0 \\ u'_0 \end{bmatrix}$$

Conjugate matrix

Like it is one big lens
Example: 2 thin lenses in contact

\[
\begin{bmatrix}
y_i \\
u_i
\end{bmatrix} = T_i R_2 R_1 T_o = \begin{bmatrix}
1 & t_i \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 1 \\
-1/f_2 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
-1/f_1 & 1
\end{bmatrix} \begin{bmatrix}
1 & t_o \\
0 & 1
\end{bmatrix} \begin{bmatrix}
y_o \\
u_o
\end{bmatrix}
\]

\[
M = \begin{bmatrix}
1 & 0 \\
-(\varphi_1 + \varphi_2) & 1
\end{bmatrix}
\quad N = \begin{bmatrix}
1-t_i(\varphi_1 + \varphi_2) & t_o + t_i - \frac{t_i t_o}{f_2} - \frac{t_i t_o}{f_1} \\
-(\varphi_1 + \varphi_2) & -\frac{t_o}{f_2} - \frac{t_o}{f_1} + 1
\end{bmatrix}
\]
Matrices Summary

Free Space Propagation

\[ M = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}. \]

Refraction at spherical interface

\[ M = \begin{bmatrix} \frac{1}{n_2 - n_1} & 0 \\ \frac{n_1}{n_2 R} & \frac{n_1}{n_2} \end{bmatrix}. \]

Convex, \( R > 0 \); concave, \( R < 0 \)

Refraction at interface

\[ M = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}. \]

Thin Lens

\[ M = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}. \]

Convex, \( f > 0 \); concave, \( f < 0 \)
Matrices Summary

Reflection from plane mirror

\[ M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

Cascaded optical components

\[ M = M_N \cdots M_2 M_1 \]

Reflection from spherical mirror

\[ M = \begin{bmatrix} \frac{1}{R} & 0 \\ 0 & 1 \end{bmatrix} \]

Set of parallel plates

\[ M = \begin{bmatrix} 1 & \sum_{i=1}^{N} \frac{d_i}{n_i} \\ 0 & 1 \end{bmatrix} \]

Concave, \( R < 0 \); convex, \( R > 0 \)
Properties of M and N

\[ |M| = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD - BC = 1 \]

\[ |R| = |T| = |M| = |N| = 1 \]

Write out the matrix equation for N:

\[ y_{K+1} = N_{11}y_0 + N_{12}u_0' \]

\[ u_{K+1} = N_{21}y_0 + N_{22}u_0' \]
Properties of M and N

If planes 0 and K+1 are conjugates, final ray height does not depend on initial ray angle:

\[ N_{12} = 0 \]  \hspace{1cm} \text{Conjugate condition}

If plane 0 is the object space focal plane, the slope at the exit plane depends only on the object height:

\[ N_{22} = 0 \]  \hspace{1cm} \text{Object at front focal plane}

If plane K+1 is the image space focal plane, the image-space ray height depends only on the entrance angle:

\[ N_{11} = 0 \]  \hspace{1cm} \text{Image at rear focal plane}

If the system is afocal, the direction of the image-space ray depends only on the direction of the object-space ray:

\[ N_{21} = 0 \]  \hspace{1cm} \text{Afocal condition}
Using M and N

Find image plane given object

Conjugate planes so $N_{12} = 0$
Using M and N matrices

\[ N = T_{K+1} M T_0 \]

\[
= \begin{bmatrix}
1 & d'_K \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
1 & -d_1 \\
0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
A + d'_K C & B + d'_K D - d_1 (A + d'_K C) \\
C & D - d_1 C
\end{bmatrix}
\]

\[
= \begin{bmatrix}
A + d'_K C & 0 \\
C & D - d_1 C
\end{bmatrix}
\]

Conjugate condition

\[ d'_K = -\frac{d_1 A - B}{d_1 C - D} \]

\[ N_{12} = 0 \text{ gives the image location} \]

E.g. single lens

\[ d'_K = -\frac{d_1 1 - 0}{d_1 (-\phi) - 1} \Rightarrow \frac{1}{d'_K} = \frac{1}{d_1} + \phi \]
Form of $N$

Effective Focal length of system or thick lens

\[ M \equiv \frac{y_{K+1}}{y_0} = N_{11} = A + d'_K C \]

If $N_{12} = 0$ then $N_{11}$ is the magnification

\[ N_{22} = \frac{1}{M} \]

Determinant $= 1$

\[ F = \frac{1}{\Phi} \equiv \frac{y_0}{-u_{K+1}} \]

Effective focal length & system power
Form of $N$

Effective Focal length of system or thick lens

$u_{K+1} = N_{21} y_0 + N_{22} u_0$

$N_{21} = -\Phi$

$N = \begin{bmatrix} M & 0 \\ -\Phi & 1/M \end{bmatrix}$

$E.g.\ single\ lens$

$N = T_1 R_1 T_0$

$= \begin{bmatrix} 1-\phi t' & tt'(-\frac{1}{t} + \frac{1}{t'} + \phi) \\ -\phi & 1+\phi t \\ \end{bmatrix}$

$= \begin{bmatrix} \frac{t'}{t} & 0 \\ -\phi & \frac{1}{t'} \end{bmatrix}$
2 lens separated by $d$

\[ \begin{bmatrix} y_i \\ u_i \end{bmatrix} = T_4 R_3 T_2 R_1 T_0 \begin{bmatrix} y_o \\ u_o \end{bmatrix} \quad M = R_3 T_2 R_1 \quad \text{system matrix} \]

\[ M = \begin{bmatrix} 1 - d/f_2 & d \\ -1/f_1 + d/f_1 f_2 - 1/f_2 & -d/f_1 + 1 \end{bmatrix} \]

\[ -M_{21} = \frac{1}{f_{\text{eff}}} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} \]
Question

Effective F of 4F system (d=f1+f2) ?

\[-M_{21} = \frac{1}{f_{\text{eff}}} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}\]

It is afocal

d> f1+f2, f is negative

d< f1+f2, f is positive
Periodic Systems

\[
\begin{bmatrix}
y_m \\
\theta_m \\
\end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^m \begin{bmatrix} y_0 \\
\theta_0 \\
\end{bmatrix}.
\]

We can also apply the relations

\[
y_{m+1} = Ay_m + B\theta_m \\
\theta_{m+1} = Cy_m + D\theta_m
\]

Derive equations that determine the evolution of \( y \), get rid of slope for above:

\[
\theta_m = \frac{y_{m+1} - Ay_m}{B}.
\]

Replacing \( m \) with \( m + 1 \)

\[
\theta_{m+1} = \frac{y_{m+2} - Ay_{m+1}}{B}.
\]

Substitute these into equations above:
Periodic Systems

Yields:

\[ y_{m+2} = 2by_{m+1} - F^2 y_m, \]

Recurrence Relation

Where,

\[ y_m = y_0 h^m, \]

\[ b = \frac{A + D}{2} \]

\[ F^2 = AD - BC = \text{det}[M], \]
Periodic Systems

Guess a solution with initial conditions $y_0$ and slope $\theta_0$ such that:

Put solution into recurrence relation results in:

$$y_m = y_0 h^m,$$

For $m$ trip around

$$h^2 - 2bh + F^2 = 0,$$

$$h = b \pm j(F^2 - b^2)^{1/2}.$$  

Can define

$$\varphi = \cos^{-1} \frac{b}{F}$$

If $n1 = n2$, then det $F = 1$

Solution can be rewritten as:

$$y_m = y_{\text{max}} F^m \sin(m \varphi + \varphi_0)$$

Where

$$y_{\text{max}} = \frac{y_0}{\sin \varphi_0}$$
Periodic Systems

\[ y_m = y_{\text{max}} F^m \sin(m\varphi + \varphi_o) \]

For the ray trajectory to be stable, \(|b| \leq 1\)

Case det \(F=1\)

Or

\[ \frac{|A + D|}{2} \leq 1 \]
Why did we waste this time covering periodic optical systems?

- How about resonators -> laser cavities and etalons? They can be considered periodic optical systems. Laser cavities have to have stable ray paths for the laser to function...
- This will be a homework problem and is very important example...
Try this

Use a ray trace to find the image location and magnification of this system by tracing a single axial ray (think about how to get both those pieces of information from this ray).

![Diagram]

<table>
<thead>
<tr>
<th>Surface $k$</th>
<th>System $\phi_k = 1/f$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$d_k'$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Axial ray</td>
<td>$y_k$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$u_k'$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Reading

W. Smith “Modern Optical Engineering”

Chapter 3 and Chapter 4