Advanced Paraxial Design

• Phase Space
• Delano Diagrams
• Modify equations
• Example
Phase space

At the object

\[ f = \frac{\sin \theta}{\lambda} \]

Resolvable spots

\[ N = \frac{2}{\lambda} \times H \]

\[ N = 1000 \]

\[ H = n_k u_k y_k - n_k \overline{u_k} \overline{y_k} \]

\[ = \frac{1}{10} \times 2.5 - 0.0 \]

\[ = 0.25 \]
Phase space

Before the lens

\[ H = n_k u_k \bar{y}_k - n_k \bar{u}_k y_k \]
\[ = \frac{1}{10} \cdot 2.5 - 0.75 \]
\[ = 0.25 \]

Resolvable spots
\[ N = \frac{2}{\lambda} \cdot H \]
\[ N = 1000 \]
Phase space

After the lens

\[ [x', f'] = [x, f - x/(F\lambda)] \]

\[
H = n_k u_k \bar{y}_k - n_k \bar{u}_k y_k
= \frac{-1}{20} 2.5 - \frac{-1}{20} \cdot 7.5
= -.125 + .375
= .25\]
Phase space
At the Fourier plane

\[ [x', f'] = [x + z\hat{f}, f] \]

\[ H = n_k u_k y_k - n_k u_k y_k \]

\[ = \frac{-1}{20} - \frac{-1}{20} \cdot .5 \]

\[ = .25 \]
Phase space
At the Image plane

\[ [x', f'] = [x + z\lambda f, f] \]

\[ H = n_k u_k \bar{y}_k - n_k \bar{u}_k y_k \]
\[ = \frac{-1}{20} (-5) - \frac{-1}{20} \cdot 0 \]
\[ = 0.25 \]

Resolvable spots
\[ N = \frac{2}{\lambda} \cdot H \]
\[ N = 1000 \]
Delano diagrams (y-y diagrams)
A paraxial \textit{design} tool

- y-u tracing calculates ray heights through system given lens prescription.

- The Delano diagram calculates the lens prescription given the ray heights.

- The optical invariant H is a scale factor for the entire calculation and thus all lens spacings and powers and the locations of pupils, principal planes and focal planes can be adjusted to give a desired resolvable number of spots.
Delano diagrams
A graphical tool for system design

Also known as y-ybar diagrams since we will plot the height of the **marginal ray** (y) and the height of the **chief ray** (ybar) as the two rays evolve through the system.

D_{obj} = 2.5 \text{ mm} \\
f = 50 \text{ mm} \\
D_{AS} = 25 \text{ mm} \\
D_{img} = 5 \text{ mm}

To visualize the approach, consider putting the **chief ray** in the x-z plane while leaving the **marginal ray** in the y-z plane.

Note that, by convention, the object height is negative.
Delano diagrams
Introduction, single lens system

Now consider a skew ray whose x coordinate is given by the chief ray and whose y coordinate is given by the marginal ray. That is, the chief and marginal rays are the projections of this skew ray in the xz and yz planes.

Now we look at this skew ray from the image plane (that is, we project the ray onto the xy plane: The ray spirals clockwise (and can never do otherwise).
Afocal system
Keplerian telescope

Paraxial Ray diagram

Delano Diagram
Delano diagrams
Lenses on the diagram

Lenses bend the skew ray. In general, the ray continuously spirals clockwise.

Negative lens, virtual image

Positive lens, virtual image
Delano diagrams

Lenses on the diagram

Lenses bend the skew ray. In general, the ray continuously spirals clockwise.

Positive lens, real image

Not allowed
Delano diagrams

Conjugate line

For ease of drawing, let's momentarily draw the chief ray back in the yz plane so we have a 2D drawing. Note that we have kept the negative object height convention. Now let us shift the object to a new object location which is conjugate to a new image location. But, don't draw a new marginal or chief ray.
Delano diagrams

Conjugate line

Note that the “old” marginal or chief rays cross both the "new" object and "new" image. At the new conjugate planes, there is a new magnification and the heights of the old marginal and chief rays at the new object (unprimed) and image (primed) planes should be linearly related:

\[
\frac{y}{y'} = \frac{\bar{y}}{\bar{y}'}
\]

The new image location can thus be found via a "conjugate line" drawn through the origin. The new magnification is:

\[
m = \frac{\bar{y}'}{\bar{y}} = \frac{y'}{y} = \frac{"\text{old" ray heights at \"new" image location}}{"\text{old" ray heights at \"new" object location}}
\]
How to find distances

Since we projected out the z dimension, it seems unlikely that the Delano diagram can tell us about z locations of elements. It turns out that it can, with a little manipulation.

Let’s start by finding the reduced distance between elements in terms of y and ybar:

\[ y_{k+1} = y_k + u_k' d_k' \]  
Marginal ray transfer eq.

\[ \bar{y}_{k+1} = \bar{y}_k + \bar{u}_k' d_k' \]  
Chief ray transfer eq.

Combine the two equations

\[ y_{k+1} \bar{y}_{k+1} = y_k \bar{y}_k + u_k' d_k' \bar{y}_k + \bar{u}_k' d_k' y_k = y_{k+1} \bar{y}_{k+1} \]

and solve for the reduced distance:

\[ d_k' = \frac{y_k \bar{y}_{k+1} - y_{k+1} \bar{y}_k}{\bar{u}_k' y_{k+1} - \bar{u}_k' \bar{y}_{k+1}} H \]

Multiply this by \( n_k \) to get actual distance. So if we know \( H \), y and ybar give the element separation. How would we specify \( H \)?

Remember that at an object or image plane, \( y=0 \), y limits the field and \( u \) is the numerical aperture. (Note \( H<0 \) since \( ybar<0 \)) Then:

\[ H = nu \bar{y} = NA \frac{L}{2} = .6 \frac{\lambda}{\chi} \frac{L}{r_0} \approx \frac{\lambda}{2} N \text{spots} \]
Calculation of distances

from the y-ybar diagram

Define the **skew ray** position vector in the xy plane at a surface $k$

$$\vec{r}_k = \bar{y}_k \hat{x} + y_k \hat{y}$$

Any two such points and the origin form a triangle. The area of that triangle can be found by:

$$\vec{A}_{k,k+1} = \frac{1}{2} \vec{r}_k \times \vec{r}_{k+1} = \frac{1}{2} \left[ y_{k+1} \bar{y}_k - y_k \bar{y}_{k+1} \right] \hat{z}$$

Comparing to the equation for separation distance, we find

$$d_{k,k+1} = \frac{2n_k A_{k,k+1}}{H}$$

So $d$ between $k$ and $k+1$ is given by $A_{k,k+1}$.
Calculation of distances
from the y-ybar diagram

\[ d_{k,k+1} = \frac{2n_k A_{k,k+1}}{H} \]

In our example

\[ d_{0,1} = \frac{2(1)\frac{1}{2}[12.5(-2.5)]}{(12.5/75)(-2.5)} = \frac{-31.25}{-0.416667} = 75 \]

Note that both A and H are negative since \( \vec{r}_0, \vec{r}_1, (0,0) \) proceed clockwise.
This gives the initial ray angle
\[ u'_0 = \frac{y_1 - y_0}{d_{0,1}} = \frac{12.5}{75} = 0.16667 \]
How to find lens powers

We now know the ray heights and ray angles. Using a similar approach to how we found $d$, we can find the lens power:

$$u'_k = u_k - y_k \phi_k$$  \hspace{1cm} \text{Marginal ray refraction eq.}

$$\overline{u}'_k = \overline{u}_k - \overline{y}_k \phi_k$$  \hspace{1cm} \text{Chief ray refraction eq.}

Combine and solve for $f$:

$$\phi_k = \frac{\overline{u}_k u'_k - u_k \overline{u}'_k}{H}$$

Thus we have found the distance between lenses, the ray angles and the lens power just from $y$ and $\overline{y}$ at each surface. This is the essence of the Delano method.
Example system for next few pages

On the following pages, we’ll use a system roughly like this. (this is just a sketch and not guaranteed to be exactly the system on the remaining pages, but has roughly the right properties.
Finding the pupils

Use the conjugate line \( \bar{y} = 0 \)

As on “conjugate line” page, extend skew ray until it intersects the conjugate line. The intersection gives \( y \) and \( y_{\text{bar}} \) of the entrance pupil. Note that \( y_{\text{bar}} \) is zero, as it should be for the pupil.

So \( R_{\text{EntPupil}} = 10 \). If \( H = -0.2 \), then

\[
d_{1,\text{EntPupil}} = \frac{2(n_0) \frac{1}{2} [y_{\text{EntPupil}} \bar{y}_1 - y_1 \bar{y}_{\text{EntPupil}}]}{H} = \frac{2(1) \frac{1}{2} [10(-2) - 6(0)]}{-0.2} = 100
\]
Finding the pupils

So \( R_{\text{ExitPupil}} = 6 \).

\[
d_{2,\text{ExitPupil}} = \frac{2(n_0)^{\frac{1}{2}}[y_2 \bar{y}_{\text{ExitPupil}} - y_{\text{ExitPupil}} \bar{y}_2]}{H} = \frac{2(1)^{\frac{1}{2}}[4(0) - 6(1)]}{-0.2} = -30
\]
Finding the principal planes

Extend the object- and image-space rays until they cross. Note that this now looks like a single refraction (our definition of PPs).

We can draw a conjugate line through the intersection, thus we have found the conjugates of unit magnification. How do we find the locations of P and P'?
Finding the principal planes

\[ d_{1,p} = \frac{2(n_0) \frac{1}{2} [y_P \bar{y}_1 - y_1 \bar{y}_P]}{H} = \frac{2(1) \frac{1}{2} [8(-2) - 6(-1)]}{-0.2} = 50 \]

\[ d_{2,p} = \frac{2(n_0) \frac{1}{2} [y_{P'} \bar{y}_2 - y_2 \bar{y}_{P'}]}{H} = \frac{2(1) \frac{1}{2} [8(1) - 4(-1)]}{-0.2} = -60 \]
Finding front focal length

Draw the conjugate line parallel to the image-space ray. The image is now at infinity, thus the object-space point is the front focal plane.

\[ d_{1,F} = \frac{2(n_o) \frac{1}{2} [y_F \bar{y}_1 - y_1 \bar{y}_F]}{H} = \frac{2(1) \frac{1}{2} [5(-2) - 6(-2.5)]}{-0.2} = -25 \]

A conjugate line between the object space and image space with the image space intersection at infinity.
Finding back focal length

Draw the conjugate line parallel to the object-space ray. The object is now at infinity, thus the image-space point is the back focal plane.

\[ d_{2,F'} = \frac{2(n_0)\frac{1}{2} [y_{F'} - y_2 y_{F'}]}{H} = \frac{2(1)\frac{1}{2} [3(1) - 4(1.5)]}{-0.2} = 15 \]

A conjugate line between the image space and object space with the object space intersection at infinity.
Finding the effective focal length

The EFL is the distance from the front focal point $F$ to the front principal plane $P$ or the distance from the back principal plane $P'$ to the back focal point $F'$. Note that $[F, PP', F', O]$ is a parallelogram so the area of the two triangles is guaranteed to be equal therefore the front and back EFL are identical, as required.

Starting from $H$ and the height of the marginal and chief ray at each surface, we have found the complete prescription for the system including all lens sizes, locations and power, the locations of pupils, principal planes and the effective focal length.

$$F = d_{F,P} = \frac{2(n_0)\frac{1}{2}[y_P \bar{y}_F - y_F \bar{y}_P]}{H} = \frac{2(1)\frac{1}{2}[8(-2.5) - 5(-1)]}{-0.2} = 75$$