Interference

• Principle
  • coherence
  • wavefront division interferometers
  • amplitude division interferometers

• Holography
  • thin and thick holograms
Interference

Basics

To interfere (completely) two electric fields must be at the

\[ I = |\vec{E}_1 + \vec{E}_2|^2 = |E_1|^2 + |E_2|^2 + 2|E_1||E_2|\cos \left[ (\omega_1 - \omega_2)\tau - (\vec{k}_1 - \vec{k}_2) \cdot \vec{r} + (\phi_1 - \phi_2) \right] \]

Summation of intensities \hspace{1cm} Interference

When don’t you see interference fringes?

\[ \Delta t \geq \frac{1}{\Delta f} \hspace{1cm} \text{Integration time exceeds coherence time} \]

\[ \Delta r \geq \frac{2\pi}{\Delta k} \hspace{1cm} \text{Path length difference exceed coherence length} \]

Then \( I = I_1 + I_2 \)
Interferometers
Wavefront division interferometry

What is $E_{\text{tot}} (\vec{r}')$ given N identical apertures which radiate an electric field $E_{\text{ap}} (\vec{r})$?

$$E_{\text{tot}} (\vec{r}') = \sum_{n=1}^{N} E_{\text{ap}} (\vec{r}' - \vec{r}_n)$$

$$= \int E_{\text{ap}} (\vec{r}' - \vec{r}) A (\vec{r}) d\vec{r}$$

$$A (\vec{r}) = \sum_{n=1}^{N} \delta (\vec{r} - \vec{r}_n)$$

A is the “array function”

We know that the radiation in the Fraunhofer regime (angular spectrum at infinite distance or in back focal plane of lens) is given by the Fourier transform of the electric field in the near field:

$$E_{\text{tot}} (\vec{k}) = \mathcal{F} \left\{ E_{\text{tot}} (\vec{r}') \right\}$$

$$= E_{\text{ap}} (\vec{k}) A (\vec{k})$$

Convolution in space becomes multiplication in k-space

**ARRAY THEOREM:** Diffraction from array is diffraction from aperture multiplied by the FT of the array distribution.
Example

Single and double slits

\[
E_{\text{tot}} (\vec{r}') = \text{Sinc} \left( \frac{k_x \frac{L}{2}}{f} \right) = \text{Sinc} \left( \frac{\pi x L}{\lambda f} \right)
\]

\[
E_{\text{tot}} (\vec{r}') = \text{Sinc} \left( \frac{k_x \frac{L}{2}}{f} \right) \cos \left( \frac{k_x \frac{d}{2}}{f} x \right) = \text{Sinc} \left( \frac{\pi L}{\lambda f} x \right) \cos \left( \frac{\pi d}{\lambda f} x \right)
\]
Amplitude division
Finite-impulse response interferometers

\[ I = |\vec{\delta}_1 + \vec{\delta}_2|^2 = |E_1|^2 + |E_2|^2 + 2|E_1||E_2|\cos \left( \frac{2\pi\nu c}{\lambda S} \right) \]

\[ \nu = \frac{c}{\Delta S} \]

Free-spectral range
Partial coherence

What if the light is not purely monochromatic? Now we must explicitly include the finite observation time:

\[
I = \int |\bar{\delta}_1 + \bar{\delta}_2| \, dt
\]

\[
= \Gamma_{11}(0) + \Gamma_{22}(0) + 2 \sqrt{\Gamma_{11}(0) \Gamma_{22}(0)} \Re \left[ \frac{\Gamma_{12}(\tau)}{\sqrt{\Gamma_{11}(0) \Gamma_{22}(0)}} \right]
\]

where

\[
\Gamma_{ij}(\tau) = \int E_i(t) E_j^*(t + \tau) \, dt
\]

Relative delay between two arms of \(\tau = \Delta S/c\)

Example: Two lasers differing in FSR by 10%:

Coherence function, \(\Gamma\)
Partial coherence

\[ \gamma_{12} = \frac{\Gamma_{12}(\tau)}{\sqrt{\Gamma_{11}(0)\Gamma_{22}(0)}} \]

Complex degree of mutual coherence

\begin{align*}
|\gamma_{12}| &= 1 & \text{coherent limit} \\
0 < |\gamma_{12}| &< 1 & \text{partially coherence} \\
|\gamma_{12}| &= 0 & \text{incoherent limit}
\end{align*}
Amplitude division

Infinite-impulse response interferometers

Fabry-Perot interferometer, aka “Etalon”

\[ S = 2 \, n \, d \, \cos \, \theta \]

\[ E_{\text{trans}} = E_{\text{inc}} \left[ t t + t r r t + j k \, _{0} \, S + t r r r r t - j k \, _{0} \, 2 \, S + \ldots \right] \]

This is a geometric series. Writing \( R = |r|^2 \), \( T = |t|^2 \)

\[ T = \frac{I_{\text{trans}}}{I_{\text{inc}}} = \frac{1}{1 + F \, \sin^2 \left( \frac{\pi}{\nu} \right)} \]

Finesse

\[ F = \frac{4 \, R}{\left(1 - R\right)^2} \]

\[ \nu_0 = \frac{c}{S} = \frac{c}{2 \, n \, d \, \cos \, \theta} \]
As you change the length of the cavity the frequency that is transmitted changes (homework)

High F can result in very high resolution frequency scans.

10’s of MHz resolution
Holography

- Recording medium must physically respond to intensity (not field)
- We wish to record the complete optical field
- Interference:

\[
I_{\text{WRITE}} = |E_O + E_R|^2
= |E_O|^2 + |E_R|^2 + E_O^*E_R + E_OE_R^*
\]

\[
E_{\text{READ}} = T E_R
\propto \left[ |E_O|^2 + |E_R|^2 + E_O^*E_R + E_OE_R^* \right] E_R
= \left[ |E_O|^2 + |E_R|^2 \right] E_R + E_O^*E_R E_R + E_O|E_R|^2
\]

Interfere object with reference

Film transmission $\alpha$ to $I_{\text{WRITE}}$
Recording Materials

- Film
- Dichromatic gelatin
- Photopolymers
- Photorefractive crystals
- Photorefractive polymers
- Photochromatic glasses/polymers
**Conventional photopolymer media**

**Mechanism**

System consists of monomers dissolved in a matrix.

Holographic exposure produces a spatial pattern of photoinitiated polymerization.

Concentration gradient in unreacted monomers induces diffusion of species.

Diffusion produces a compositional gradient, establishing a refractive index grating (Δn).
Recording Process for Photopolymers

Precure of Media

Recording

Post cure

Bleach

Media is designed with to include inhibitor to preserve shelf life. Precure consumes the inhibitor.

Metered exposures translates the optical interference patterns into refractive index patterns in the media.

Post cure consumes any unreacted active recording components in the media.

Bleach consumes the photoinitiator, the light trigger of the media.
Holography

- The bandwidth of \( E_o \) equal to \( B_o \) and the bandwidth of \( E_R \) equal to \( B_R \)
- The bandwidth of \( |E_o(r)|^2 \) is equal to the bandwidth of \( E_o(k) \ast E_o(k) \) which is \( 2B_o \)

\[
-(k_o - k_R) \quad T(k) \quad k_o - k_R
\]

\[
\text{Conj} \quad \text{Ambiguity} \quad \text{Obj}
\]

\[
B_o + B_R \quad \text{Max}[2B_o, 2B_R] \quad B_o + B_R
\]

\[
E_{READ}(k) = T(k) \ast E_R(k)
\]

\[
-(k_o - k_R) \quad k_o - k_R
\]

\[
\text{Conj} \quad \text{Ambiguity} \quad \text{Obj}
\]

\[
B_o + 2B_R \quad B_R \text{ Max}[2B_o, 2B_R] \quad B_o + 2B_R
\]

\[
k_o - k_R > \frac{1}{2} \left(B_R \text{ Max}[2B_o, 2B_R] + B_o + 2B_R \right) \quad \text{To separate terms}
\]

\[
k_o - k_R > \frac{3}{2} B_o \quad \text{Special case for plane-wave reference (B_R=0)}
\]
Phase Conjugate Readout

Time reversal

Write

Reference

Object

Conjugate read

Conjugate reference

Illuminate with phase-conjugate reference

\[ E_{\text{CONJ}} = T E_R^* \]

\[ \propto \left[ |E_O|^2 + |E_R|^2 + E_O^* E_R + E_O E_R^* \right] E_R^* \]

\[ = \left[ |E_O|^2 + |E_R|^2 \right] E_R + E_O^* E_R \left| E_R \right|^2 + E_O E_R^* E_R \]
Phase Conjugate Readout
Time reversal

Use in imaging through phase (not amplitude!) perturbation:

Write

Ref

Object

Phase perturbation

Read

Reconstruction

Ref
Holography (thin hologram)

Readout with Tilted Reference

- Add spatial frequency to reconstruction at plane of hologram. (conservation of momentum)
- Similar to tilt – precisely equivalent to grating deflection.
- Only limitation is TIR = wide angular bandwidth.

\[ E_{READ} = T \left| E_R \right|^2 e^{-jk_x x} \]
\[ \propto \left[ \left| E_O \right|^2 + \left| E_R \right|^2 \right] E_R e^{-jk_x x} + E_R^* e^{jk_x x} E_R E_R + E_O e^{-jk_x x} \left| E_R \right|^2 \]
Holography (thin hologram)

Readout with different $\lambda$

Wide spectral changes possible; however....
Huge chromatic aberrations for large changes in $\lambda$
Holography

Recording plane, shown reflective hologram

**Fourier plane**
- Spatial filters
- Sensitive to flatness
- Insensitive to amplitude

**Image plane**
- Security images
- Insensitive to flatness
- Sensitive to amplitude
- Curved recording ref & white-light, plane read ref yields “rainbow hologram”

**Fresnel plane**
- 3D imagery
- Intermediate properties
Thick vs Thin Holograms

Holograms may be written in either the thick or thin regime. Thin holograms, such as those written on ordinary photographic film, effectively confine their diffractive interaction to a single plane and are not suitable for dense multiplexing. One basic criteria for defining holograms as thin or thick is the $Q$ parameter suggested by Klein and Cook, and a grating strength parameter $[\text{i}, \text{ii}]$. These parameters determine whether the grating is in the Bragg (single order is diffracted) or Raman-Nath (multiple orders are diffracted) diffraction regimes. The $Q$ parameter is given by

$$ Q = \frac{2\pi\lambda}{L/n_0\Lambda^2} \quad F = 16n^2\sin^2\theta/e_r $$

where $\lambda$ is the vacuum wavelength of the light, $L$ is the thickness of the recording layer, $n_0$ is the nominal refractive index of the medium, and $\Lambda$ is the grating period. If $Q < 1$ and the grating has a large enough index change (equivalent to a few percent diffraction efficiency in thin media) then the gratings are considered in the thin media or Raman-Nath regime. For values of $Q > 1$, the gratings are considered to be thick, volume gratings in the Bragg diffraction regime. The presence of many grating planes or grating periods supports Bragg diffraction from these structures.


**Bragg Selectivity of Thick Holograms**

Path difference $\Delta d = 2 \Lambda \sin \theta$
Phase difference $\Delta \phi = (2 \pi / \lambda) \times \Delta d$

- **When** $\theta = \theta_o$ (used to record), $\Delta \phi = 2\pi$, adds up in phase
- **Else**, $\Delta \phi = 2\pi \Lambda \Delta \theta \cos(\theta_o) / \lambda$

Sum waves from 0 to $N_g = L \tan (\theta_o) / \Lambda$
First null $\Rightarrow N_g \times \Delta \phi = 2\pi$

$\Lambda = \lambda / 2 \sin(\theta_o)$

Thickness determines width of Bragg peak

$\Delta \theta = \lambda / (2 \sin(\theta_o) L)$ first null
Kogelnik’s Coupled Wave Analysis

Kogelnik’s famous equations for plane wave holograms. For a pure index grating in transmission geometry the diffraction efficiency is given by

\[ \eta = \sin^2 \left( \nu + \xi^2 \right) / \left( 1 + \frac{\xi^2}{\nu^2} \right), \]

where the parameters are given by

\[ \nu = \frac{\pi n_1 L}{\lambda \left( \cos^2 \theta - \frac{K \cos \theta \cos \phi}{\beta} \right)^{1/2}}, \]

and

\[ \xi = (\Delta \theta \cdot K \sin(\phi - \theta_0) - \Delta \lambda K^2 / 4\pi n_0) + 2(\cos \theta - \frac{K \cos \phi}{\beta}). \]

\( \eta \) is the diffraction efficiency, \( K \) is the grating number, \( (2\pi/\Lambda) \), where \( \Lambda, \lambda, \) and \( L \) are period, wavelength and media thickness. \( \theta \) is the angle of the reference beam (outside the media) on readout measured from media normal, \( n_0 \) is the bulk index of the material, \( n1 \) is the index perturbation, \( \phi \) is the grating slant angle inside the material measured from normal and \( \beta \) is \( 2\pi(\varepsilon_o)^{1/2}/2\lambda \) where \( \varepsilon_o \) is the bulk dielectric constant. The parameters \( \Delta \lambda \) and \( \Delta \theta \) are the deviations from the Bragg conditions for wavelength and angle respectively.

• The functional form for reflective holograms is very different. The Kogelnik equations for a pure index grating in reflection are given by

\[ \eta = 1 / \left\{ 1 + \left( 1 - \frac{\xi^2}{\nu^2} \right) / \sinh^2 \left( \nu^2 - \xi^2 \right)^{1/2} \right\}, \]

where \( \nu \) is equal to equation above but multiplied by \( j \), and \( \xi \) is equal to the negative of equation above due to the fact that the angle of the signal is negative in this geometry.

Bragg Selectivity
K-Space view

- **Diffraction efficiency:**
  - Function of reference beam angle.
  - Dependent on media thickness
  - Sinc function is due to finite thickness of media.

\[ DE \propto \text{sinc}^2\left(\frac{\Delta k L}{2\pi}\right) \]

Media Thickness

\[ \Delta k_z \]
Transmission vs Reflection

- Transmissive gratings come off the k-sphere quickly with angle change

\[ \Delta \theta_{\text{trans}} = \frac{\lambda \cos \theta_s}{L \sin(\theta_r + \theta_s)} \quad \Delta \lambda_{\text{trans}} = \frac{\lambda^2 \cos \theta_s}{L(1 - \cos(\theta_R + \theta_s))} \]

- Reflective gratings drag along the k-sphere with angle change and therefore have poor angle selectivity but high wavelength selectivity

\[ \Delta \theta_{\text{ref}} \approx \sqrt{\frac{\lambda}{L}} \]

\[ \left( \Delta \lambda \right)_{\text{Ref}} = \frac{\lambda^2 \cos \theta_s}{2L \sin^2 \frac{1}{2}(\theta_r + \theta_s)} . \]

Pure reflection
Angle Multiplexing

Each Page has unique address “angle” within a book

Reference Beams

Signal beam

Media

Camera

SLM

Inverse read-out/data recovery

SLM data page of 1.4 mega pixels
Measurements of Dynamic Range (M/ #)

Angle Multiplexed Holograms

Measure of Dynamic Range: M/#

\[
M/# = (\text{Number of Holograms}) \times (\text{Ave. Diff. Eff.})^{1/2}
\]

\[
M/# \sim \Delta n \times (\text{thickness})
\]
## 10 Multiplexing Techniques

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<tr>
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</tbody>
</table>
Shift Multiplexing

\[ \Delta x = \frac{\lambda d}{2L \sin(\Theta/2)} + \frac{\lambda}{2(NA)} \]

Uncertainty in grating (standard angle mux)  Uncertainty in position

Concept similar for infinite and finite number of plane waves
Data page holograms

Fourier geometry:
- Each SLM pixel can be treated as a Plane Wave.
- Pixels result in a manifold of grating vectors.
Temperature Effects on Plane Wave Holograms

Two Temperature Effects:

1) Media Expansion
2) Media Index of Refraction Change.

\[ \vec{K}_G \to K_{Gx} \hat{x} + K_{Gy} \hat{y} + \frac{K_{Gz}}{z_{sf}} \hat{z} \]

\[ n \to n - \Delta n \]

These effects change the angles of both beams on hologram recovery.
Data page holograms and Fourier geometry

Temperature affects each pixel grating, changing the manifold curvature.

- Impossible to Bragg match the entire data page.
- Only a narrow band will be recovered on the camera

\[
\tilde{K}_G \rightarrow K_{Gx}\hat{x} + K_{Gy}\hat{y} + \frac{K_{Gz}}{z_{sf}}\hat{z}
\]

\[
n \rightarrow n - \Delta n
\]
Wavelength Compensation

- Wavelength change will compensate new manifold curvature.
- Reference beam angle must be changed for new wavelength
- Recovered data page is magnified:
  - Zoom lens or Over-sampled detection required (M. Ayres et al. ISOM 2005).
Determining the Wavelength Detuning

• The reference beam angle changes recovered strip location.

• Rate that “strip” moves across the camera is proportional to the amount of wavelength detuning.
Computer Generated Holograms

Calculate the hologram and print

• Sample the object and the hologram
  – How many points and at what frequency?

• Calculate the discrete Fresnel or Fourier transform on the object fields
  – Usually fast Fourier Transform used (FFT)

• Print out the resulting fields onto a transparency (our display device)
How many points?

- Fourier plane needs fewer samples than Fresnel plane hologram due to quadratic phase term adding to the bandwidth of the signal.
- For a Fourier plane hologram using a lens with focal length of \( f \), and the size of the object is \( L_\xi \times L_\eta \). Then by the Whittaker-Shannon sampling theorem, the spectrum in the hologram plane is a rectangle with dimensions

\[
2B_x = \frac{L_\xi}{\lambda f} \quad 2B_y = \frac{L_\eta}{\lambda f}
\]

and the sampling spacing required is

\[
\Delta x = \frac{1}{2B_x} = \frac{\lambda f}{L_\xi} \quad \Delta y = \frac{1}{2B_y} = \frac{\lambda f}{L_\eta}
\]

- If the FT size is \( L_x \times L_y \) then the number of samples is given by:

\[
N_x = \frac{L_x}{\Delta x} = \frac{L_x L_\xi}{\lambda f} \quad N_y = \frac{L_y}{\Delta y} = \frac{L_y L_\eta}{\lambda f}
\]
Calculation

- Calculate FT hologram as

\[
U_h(p\Delta x, q\Delta y) = \sum_{m=0}^{N_x-1} \sum_{n=0}^{N_y-1} U_o(m\Delta\xi, n\Delta\eta) \exp \left[ j2\pi \left( \frac{pm}{N_X} + \frac{qn}{N_Y} \right) \right]
\]

- This is just FFT if \( N_x \) and \( N_y \) constrained to powers of 2.

- Advantageous to add random phase at each point (like a diffuser is present). This spread the energy out and makes the hologram more efficient and reduces coherence effects.
Several methods are used. Oldest and best known method is called the Detour phase hologram.

Divide hologram cell of width \( w \) which are one full period of phase of the incident phase function.

Make area that passes light (subcell) proportional to the desired amplitude.

The position of the subcell is shifted to represent the phase desired at that point.

If the size of the width of the image \( L_u \) is sufficiently small

\[
U_f(u, v) = \sum_{p=0}^{N_x-1} \sum_{q=0}^{N_y-1} (w_X)_{pq}(w_Y)_{pq} e^{-j2\pi\frac{(\delta_x)_{pq}}{\Delta x}} \exp \left[ j\frac{2\pi}{\lambda f} (up\Delta x + vq\Delta y) \right]
\]

Which has correct amp and phase if

\[
-\frac{(\delta_x)_{pq}}{\Delta x} = \frac{\phi_{pq}}{2\pi}
\]
Example

From Goodman, *Introduction to Fourier Optics*