ECEN 5645
Introduction to Optoelectronics
Class Meeting 15
Gradient (Parabolic)
Index Multimode Fiber
Today’s Topics: Alpha (including parabolic) Index

- Step, parabolic and alpha index fiber
- Problem 2.10
- How to solve for the actual modes
- Simplified hybrid modes
- Problem 2.11
- LP modes, H-G, L-G and even I-G
The step index optical fiber. The central region, the core, has greater refractive index than the outer region, the cladding. The fiber has cylindrical symmetry. The coordinates $r$, $\phi$, $z$ are used to represent any point $P$ in the fiber. Cladding is normally much thicker than shown.
Gradient Index Fiber

• Step index guides date back to Colladon and possibly long before

• Clear telecommunications needs required a waveguide with low loss and high bandwidth

• Step index fibers are easiest to fabricate

• Using MCVD (reviewed in the next three slides) though, gradient index fiber is not much harder to fabricate than step
Making Fiber: MCVD

- In modified chemical vapor deposition, gas flows are run through the core of a preform
Sintering by Vapor Pressure (Thermophoresis)

- The increased temperature in the heat zone causes an increase in transverse velocity.
- Many thin layers are required for quality.
(a) A ray in thinly stratified medium becomes refracted as it passes from one layer to the next upper layer with lower \( n \) and eventually its angle satisfies TIR.

(b) In a medium where \( n \) decreases continuously the path of the ray bends continuously.
Graded Index (GRIN) Fiber

(a) Multimode step index fiber. Ray paths are different so that rays arrive at different times.

(b) Graded index fiber. Ray paths are different but so are the velocities along the paths so that all the rays arrive at the same time.
Fermat’s Principle of Least Time

• The path taken by rays from traveling from A to B is that which requires the least time.
• Feynman’s (actual first published by Norbert Weiner) integral over all paths shows that a wave “process” follows a path extremum.
• In a step index fiber, the least time solution is the fundamental mode.
• In a parabolic index, all rays have equal time.
Graded Index (GRIN) Fiber

The refractive index profile can generally be described by a power law with an index $\gamma$ called the profile index (or the coefficient of index grating) so that,

$$n = n_1[1 - 2\Delta(r/a)^\gamma]^{1/2} \quad ; \quad r < a,$$

$$n = n_2 \quad ; \quad r \geq a$$

Minimum intermodal dispersion

Minimum intermodal dispersion

$$\gamma_o \approx 2 + \delta - \Delta \left(\frac{4+\delta}{5+2\delta}\right) \approx 2$$

$$\sigma_{\text{intermode}} \approx \frac{n_1}{20\sqrt{3}c} \Delta^2$$
Actual Graded Profiles: The Dip

- Profiles are hard to control, especially the dip
Drawing Optical Fiber

- After MCVD, the preform is drawn
- Drawing requires uptake that requires tension
- Winding under tension will favor one direction over the other causing anisotropy
Alpha Index Profile

\[ n^2(r) = n_1^2 \left[ 1 - 2\Delta \left( \frac{r}{a} \right)^\alpha \right] \]

\[ \Delta = \frac{n_1^2 - n_2^2}{2n_1^2} = \frac{NA^2}{2n_1^2} \]

\[ \Delta |_{\text{glass}} \approx n_1 - n_2 \]

- The alpha index profile yields step (infinity), parabolic (2) and triangular (1) as cases
- Alpha of 2 is about as close to ideal as possible
Problem Set 5 Problem 2.10

- Solution by Jeremy Shuegrue
2.10 A multimode fiber

Consider a multimode fiber with a core diameter of 100 \(\mu\text{m}\), core refractive index of 1.4750, and a cladding refractive index of 1.4550, both at 870 nm. Consider operating this fiber at \(\lambda = 870\) nm.

(a) Calculate the V-number for the fiber and estimate the number of guided modes.

\[
V = \frac{2\pi}{\lambda} \left[ n_1^2 - n_2^2 \right]^{1/2} = \frac{2\pi(50\mu\text{m})}{870\text{nm}} \left[ 1.4750^2 - 1.4550^2 \right]^{1/2} = 87.414
\]

\[
M \approx \frac{V^2}{2} = \frac{(87.414)^2}{2} = 3820
\]

(b) Calculate the wavelength beyond which the fiber becomes single-mode.

\[
V_{\text{cutoff}} = \frac{2\pi}{\lambda_c} \left[ n_1^2 - n_2^2 \right]^{1/2} = 2.405
\]

\[
\lambda_c = \frac{2\pi}{2.405} \left[ n_1^2 - n_2^2 \right]^{1/2} = \frac{2\pi(50\mu\text{m})}{2.405} \left[ (1.4750)^2 - (1.4550)^2 \right]^{1/2} = 31.62\mu\text{m}
\]

(c) Calculate the numerical aperture.

\[
\text{NA} = \left[ n_1^2 - n_2^2 \right]^{1/2} = \left[ (1.4750)^2 - (1.4550)^2 \right]^{1/2} = 0.2421
\]

(d) Calculate the maximum acceptance angle.

\[
\sin \alpha_{\text{max}} = \frac{\text{NA}}{n_0}
\]

\[
\alpha_{\text{max}} = \arcsin \frac{\text{NA}}{n_0} = \arcsin \frac{0.2421}{1.0} = 14.009^\circ
\]

(e) Calculate the modal dispersion \(\Delta \tau\) and hence estimate the bit rate \(\times\) distance product.

\[
\frac{\Delta \tau}{L} \approx \frac{n_1 - n_2}{c} = \frac{1.4750 - 1.4550}{3 \times 10^8 \text{ m s}^{-1}} = 66.67 \text{ ns km}^{-1}
\]

Assuming a Gaussian light output pulse shape and that \(\Delta \tau\) contains 99.7% of the power (because of the approximation above).

\[
\sigma = \frac{\Delta \tau}{6}
\]

\[
\frac{BL}{\sigma} = \frac{0.25L}{\Delta \tau/6} = \frac{(0.25)(1)}{66.67 \text{ ns km}^{-1}/6} = 22.5 \text{ Gb s}^{-1} \text{ km}
\]
How to Solve for Mode of a Graded Cylinder

• Write harmonic Maxwell’s equations and boundary conditions in cylindrical coordinates
• Assume modal solution in curl relations
• Solve curl relations for transverse fields in terms longitudinal fields
• Find wave equations for longitudinal fields
• Note that the equations for the longitudinal fields are coupled
• One can express the tangential fields in terms of the longitudinal fields and the gradients of the index
Time Harmonic Maxwell’s Equation

\[ \nabla \times \mathbf{E}(\mathbf{r}) = j\omega \mu_0 \mathbf{H}(\mathbf{r}) \]

\[ \nabla \times \mathbf{H}(\mathbf{r}) = j\omega \varepsilon_0 n^2(x, y) \mathbf{E}(\mathbf{r}) \]

\[ \nabla \cdot \mathbf{E}(\mathbf{r}) = 0 \]

\[ \nabla \cdot \mathbf{H}(\mathbf{r}) = 0 \]

\[ \mathbf{E}(\mathbf{r}, t) = \Re \{ \mathbf{E}(\mathbf{r}) e^{j\omega t} \} \]

- Want all operations in cylindrical \((r, \theta)\) coordinates for hybrid modes
Cylindrical Coordinates

Later will write Cartesian fields in terms of cylindrical coordinates.

Description of polarization is quite different.

\[
E(r, \theta, z) = (E_r, E_\theta, E_z)
\]

\[
H(r, \theta, z) = (H_r, H_\theta, H_z)
\]
• The z components of the fields satisfy wave equations – this follows exactly (deBye)

• Transverse fields follow from longitudinal
Alpha Index Wave Equations

• The $E_z$ and $H_z$ components of the fields satisfy hopelessly coupled wave-like equations – this follows exactly (deBye)
• Transverse fields follow from longitudinal fields, but the polarization is position dependent across the core
• Can integrate the $z$ equations and plug into polarization to find hybrid modes – the modes will appear as the hybrid modes of the step index case
The polarization of these modes is especially complex and inhomogenous.
Simplified Hybrid Modes (Streifer)

- If the guidance is weak (delta<<1), the wave equations separate independent of profile
- The modes for any profile that “guides” are about the same
- Harmonic oscillators (alpha=2) are a special case of power law potentials
- The modes for the parabolic index profile in the weakly guiding limit are special as they are the harmonic oscillator modes
Step Index Wave Equations (repeat)

\[
\frac{\partial^2 \psi(r, \theta)}{\partial r^2} + \frac{1}{r} \frac{\partial \psi(r, \theta)}{\partial r} + \frac{\partial^2 \psi(r, \theta)}{\partial \theta^2} + (k_1^2 - \beta^2) \psi(r, \theta) = 0
\]

\[
\psi(r, \theta) = \begin{cases} E_z(r, \theta) \\ H_z(r, \theta) \end{cases}
\]

- The \( z \) components of the fields satisfy wave equations – this follows exactly (deBye)
- Transverse fields follow from longitudinal
Parabolic Index Wave Equations

\[
\frac{\partial^2 \psi(r, \theta)}{\partial r^2} + \frac{1}{r} \frac{\partial \psi(r, \theta)}{\partial r} + \frac{\partial^2 \psi(r, \theta)}{\partial \theta^2} + (k_1^2(r) - \beta^2) \psi(r, \theta) = 0
\]

\[
\psi(r, \theta) = \begin{cases} 
E_z(r, \theta) \\
H_z(r, \theta) 
\end{cases}
\]

\[
k_1^2(r) = k_1^2 \left[ 1 - \left( \frac{r}{a} \right)^2 \right]
\]

- The z components of the fields satisfy two dimensional harmonic oscillator equations
- SHOs have magical properties (selfoc, Talbot imaging, coherent packet formation, etc.)
Step Index Longitudinal Field Solutions

\[ \psi(r, \theta) = \begin{cases} 
J_n(\gamma r)e^{jn\theta} & \text{inside} \\
K_n(\kappa r)e^{jn\theta} & \text{outside} 
\end{cases} \]

- Boundary conditions require continuity tangential fields at \( r=a \)
- All the fields show up in each BC
- The dispersion relation four lines long
- Degeneracies appear when guidance is weak

\[ \text{Weak Guidance} \rightarrow n_1 - n_2 \ll n_2 \]
Parabolic Index Field Solutions

\[ \psi_{lm}(r, \theta) = \exp \left[ - \left( \frac{r}{2a} \right)^2 \right] \left( \frac{r}{a} \right)^m L_l^m \left[ \left( \frac{r}{2a} \right)^2 \right] \exp(i m \theta) \]

- There are no boundary conditions but there are dispersion relations that are required in order that \( l \) and \( m \) are integers
- All the fields show up in each of the two (complicated) dispersion relations
- The modes for any alpha (and mode) appear about as the step index modes
Hybrid modes with center null

- Each shape has two polarizations
- The higher the mode, the more interesting

- Modes superpose when degenerate
More Linear Combinations

- The hybrid modes here are already weakly guiding
- The linear combinations suggest there is another formulation
Problem Set 5 Problem 2.11

• Solution by
Problem 2.11  Given: \( n_{\text{water}} = 1.33 \), \( d=3\text{mm} \), \( \lambda_0= 560\text{ nm} \)

V-number: \( V=2\pi*a/\lambda*(n_1^2 -n_2^2)^{1/2} = 14757 \)
Numerical Aperture: \( NA= (n_1^2 -n_2^2)^{1/2} =0.877 \)
Acceptance angle: \( \alpha_{\text{max}}=\sin^{-1}(NA/n_0)= 41.2465^\circ \)
Total acceptance angle=\( 2*\alpha_{\text{max}} =82.49^\circ \)
Number of modes: Since \( V>>2.405 \), \( M=V^2/2 = 1.089* 10^8 \) modes
Cutoff wavelength: \( \Lambda_C= 2\pi*a*NA/2.405=3.44\text{mm} \)

The light remains guided even as the diameter of the jet increases as long as total internal reflection takes place and the incidence angle remains greater than or equal to the critical angle.
Solving for Linearly Polarized Modes for Any Weakly Guiding

- Write harmonic Maxwell’s equations using Cartesian fields in polar coordinates
- Assume modal solution in curl relations
- Solve curl relations for transverse fields in terms longitudinal fields using weakly guiding approximation where useful
- Find wave equations for longitudinal fields
- Write down the linearly polarized (LP) solutions
- Use solved curl relations to find dispersion relation
**LP Modes**

\[ E(r, \theta, z) = (E_x(r, \theta, z), E_y(r, \theta, z), E_z(r, \theta, z)) \]

\[ H(r, \theta, z) = (H_x(r, \theta, z), H_y(r, \theta, z), H_z(r, \theta, z)) \]

**Hybrid Modes**

\[ E(r, \theta, z) = (E_r(r, \theta, z), E_\theta(r, \theta, z), E_z(r, \theta, z)) \]

\[ H(r, \theta, z) = (H_r(r, \theta, z), H_\theta(r, \theta, z), H_z(r, \theta, z)) \]

- In weakly guiding fiber, LP and hybrid modes are linearly related
LP Modes

• If one assumes weakly guiding from the outset
  – The x and y components can be expressed in terms of the z-components
  – The boundary conditions separate into sets for x and y
  – The modes are either x or y polarized

• For parabolic index profiles, the LP modes are Gaussian-Hermite, Gaussian-Laguerre or elliptical modes as in a resonator
Gaussian-Hermite Modes

- These modes are products of the one dimensional solution – exact for square fibers
Comparing H-G and L-G LP modes

- Which dominate depend on the resonator/fiber, that is, ellipticity
Comparing H-G and I-G and L-G

• Which dominate depend on the resonator/fiber, that is, ellipticity but also birefringence