ECEN 5645
Introduction to Optoelectronics
Class Meeting 2

Refractive Index
And
Wave Velocity
Note On Problem Sets

- Problem sets are due on Tuesdays
- Problem sets consist of 4 problems
- Problems will be assigned to individual students to solve, outline their solution of a power point, submit to me before class on the assigned day and to discuss in class (5 minutes max).
- The assignments begin next Tuesday (9/1/15)
Today’s Topics

• **Refractive Index**
  – Phase velocity and index
  – Index as a function of wavelength
  – Sellmeier and Cauchy expansions

• **Group Velocity**
  – Wave packets
  – Group index
Refractive Index

When an EM wave is traveling in a dielectric medium, the oscillating electric field polarizes the molecules of the medium at the frequency of the wave.

The stronger is the interaction between the field and the dipoles, the slower is the propagation of the wave.
## Refractive Index

<table>
<thead>
<tr>
<th>Material</th>
<th>$\varepsilon_r$(LF)</th>
<th>$[\varepsilon_r$(LF$)]^{1/2}$</th>
<th>$n$ (at $\lambda$)</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Si</td>
<td>11.9</td>
<td>3.44</td>
<td>3.45 (at 2.15 $\mu$m)</td>
<td>Electronic bond polarization up to optical frequencies</td>
</tr>
<tr>
<td>Diamond</td>
<td>5.7</td>
<td>2.39</td>
<td>2.41 (at 590 nm)</td>
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<tr>
<td>GaAs</td>
<td>13.1</td>
<td>3.62</td>
<td>3.30 (at 5 $\mu$m)</td>
<td>Ionic polarization contributes to $\varepsilon_r$(LF)</td>
</tr>
<tr>
<td>SiO$_2$</td>
<td>3.84</td>
<td>2.00</td>
<td>1.46 (at 600 nm)</td>
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</tr>
<tr>
<td>Water</td>
<td>80</td>
<td>8.9</td>
<td>1.33 (at 600 nm)</td>
<td>Dipolar polarization contributes to $\varepsilon_r$(LF), which is large</td>
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</table>
Maxwell’s Wave Equation in an isotropic medium

\[
\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} - \varepsilon_o \varepsilon_r \mu_o \frac{\partial^2 E}{\partial t^2} = 0
\]

A plane wave is a solution of Maxwell’s wave equation

\[E_x = E_o \cos(\omega t - kz + \phi_o)\]

The phase velocity of this plane wave in the medium is given by

\[v = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon_o \varepsilon_r \mu_o}}\]

The phase velocity in vacuum is

\[c = \frac{\omega}{k_o} = \frac{1}{\sqrt{\varepsilon_o \mu_o}}\]
Phase Velocity and $\varepsilon_r$

The relative permittivity $\varepsilon_r$ measures the ease with which the medium becomes polarized and hence it indicates the extent of interaction between the field and the induced dipoles.

For an EM wave traveling in a nonmagnetic dielectric medium of relative permittivity $\varepsilon_r$, the phase velocity $V$ is given by

$$V = \frac{1}{\sqrt{\varepsilon_r \varepsilon_0 \mu_0}}$$
Refractive Index \( n \)

Phase Velocity and \( \varepsilon_r \)

\[
V = \frac{1}{\sqrt{\varepsilon_r \varepsilon_0 \mu_0}}
\]

Refractive index \( n \) definition

\[
n = \frac{c}{V} = \sqrt{\varepsilon_r}
\]
Optical frequencies

Typical frequencies that are involved in optoelectronic devices are in the infrared (including far infrared), visible, and UV, and we generically refer to these frequencies as optical frequencies.

Somewhat arbitrary range:

Roughly $10^{12}$ Hz to $10^{16}$ Hz
Low frequency (LF) relative permittivity $\varepsilon_r(LF)$ and refractive index $n$.

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Bond Energies

- An EM wave excites QM resonances
- Electronic resonances absorb in the visible and ionize in the UV – require no motion
- Vibrational resonances absorb in the N&m IR
- Rotational resonances absorb in the FIR (THz)
- Larger organic molecules may have collective resonances that extend THz/mmwave
Refractive Index and Propagation Constant

\[ k_o \quad \text{Free-space propagation constant (wave vector)} \]
\[ k_o = 2\pi/\lambda_o \]
\[ \lambda_o \quad \text{Free-space wavelength} \]
\[ k \quad \text{Propagation constant (wave vector) in the medium} \]
\[ \lambda \quad \text{Wavelength in the medium} \]

\[ n = \frac{k}{k_o} \]

In noncrystalline materials such as glasses and liquids, the material structure is the same in all directions and \( n \) does not depend on the direction. The refractive index is then isotropic.
Refractive Index and Wavelength

It is customary to drop the subscript \( o \) on \( k \) and \( \lambda \)

\[ k_{\text{medium}} = nk \]

\[ \lambda_{\text{medium}} = \frac{\lambda}{n} \]
$n$ depends on the wavelength $\lambda$

Dispersion relation: $n = n(\lambda)$

The simplest electronic polarization gives

$$n^2 = 1 + \left( \frac{N_{\text{at}} Z e^2}{\varepsilon_0 m_e} \right) \left( \frac{\lambda_o}{2\pi c} \right)^2 \frac{\lambda^2}{\lambda^2 - \lambda_o^2}$$

Sellmeier Equation

$$n^2 = 1 + \frac{A_1 \lambda^2}{\lambda^2 - \lambda_1^2} + \frac{A_2 \lambda^2}{\lambda^2 - \lambda_2^2} + \frac{A_3 \lambda^2}{\lambda^2 - \lambda_3^2}$$

$N_{\text{at}}$ = Number of atoms per unit volume

$Z$ = Number of electrons in the atom (atomic number)

$\lambda_o$ = A “resonant frequency”
Polarizability at a Resonance

- Un-broadened polarizability takes the form of a Lorentzian near a resonance
$n$ depends on the wavelength $\lambda$

Cauchy dispersion relation

\[ n = n(\nu) \]

\[ n = n_2(h\nu)^{-2} + n_0 + n_2(h\nu)^2 + n_4(h\nu)^4 \]
$n$ depends on the wavelength $\lambda$

<table>
<thead>
<tr>
<th>TABLE 1.2 Sellmeier and Cauchy coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sellmeier</td>
</tr>
<tr>
<td>-----------</td>
</tr>
<tr>
<td>SiO$_2$ (fused silica)</td>
</tr>
<tr>
<td>86.5%SiO$_2$-13.5%GeO$_2$</td>
</tr>
<tr>
<td>GeO$_2$</td>
</tr>
<tr>
<td>Sapphire</td>
</tr>
<tr>
<td>Diamond</td>
</tr>
</tbody>
</table>

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<tr>
<th>Cauchy</th>
<th>Range of $hv$ (eV)</th>
<th>$n_{-2}$ (eV$^2$)</th>
<th>$n_0$</th>
<th>$n_2$ (eV$^{-2}$)</th>
<th>$n_4$ (eV$^{-4}$)</th>
</tr>
</thead>
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<td>Diamond</td>
<td>0.05–5.47</td>
<td>$-1.07 \times 10^{-5}$</td>
<td>2.378</td>
<td>$8.01 \times 10^{-3}$</td>
<td>$1.04 \times 10^{-4}$</td>
</tr>
<tr>
<td>Silicon</td>
<td>0.002–1.08</td>
<td>$-2.04 \times 10^{-8}$</td>
<td>3.4189</td>
<td>$8.15 \times 10^{-2}$</td>
<td>$1.25 \times 10^{-2}$</td>
</tr>
<tr>
<td>Germanium</td>
<td>0.002–0.75</td>
<td>$-1.0 \times 10^{-8}$</td>
<td>4.003</td>
<td>$2.2 \times 10^{-1}$</td>
<td>$1.4 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

EXAMPLE 1.2.1  Sellmeier equation and diamond

Using the Sellmeier coefficients for diamond in Table 1.2, calculate its refractive index at 610 nm (red light) and compare with the experimental quoted value of 2.415 to three decimal places.

Solution

The Sellmeier dispersion relation for diamond is

\[ n^2 = 1 + \frac{0.3306\lambda^2}{\lambda^2 - 175 \text{ nm}^2} + \frac{4.3356\lambda^2}{\lambda^2 - 106 \text{ nm}^2} \]

\[ n^2 = 1 + \frac{0.3306(610 \text{ nm})^2}{(610 \text{ nm})^2 - (175 \text{ nm})^2} + \frac{4.3356(610 \text{ nm})^2}{(610 \text{ nm})^2 - (106 \text{ nm})^2} = 5.8308 \]

So that

\[ n = 2.4147 \]

which is 2.415 to three decimal places and m:
Some Sellmeier Plots (Kasap Table)

- SiO$_2$
- GeO$_2$
- SiGe
EXAMPLE 1.2.2  
Cauchy equation and diamond

Using the Cauchy coefficients for diamond in Table 1.2, calculate the refractive index at 610 nm.

Solution
At $\lambda = 610$ nm, the photon energy is

$$h\nu = \frac{hc}{\lambda} = \frac{(6.626 \times 10^{-34} \text{ J}s)(2.998 \times 10^8 \text{ m/s})}{(610 \times 10^{-9} \text{ m})} \times \frac{1}{1.602 \times 10^{-19} \text{ eV/J}} = 2.0325 \text{ eV}$$

Using the Cauchy dispersion relation for diamond with coefficients from Table 1.2,

$$n = n_{-2}(h\nu)^{-2} + n_0 + n_2(h\nu)^2 + n_4(h\nu)^4$$

$$= (-1.07 \times 10^{-5})(2.0325)^{-2} + 2.378 + (8.01 \times 10^{-3})(2.0325)^2$$

$$+ (1.04 \times 10^{-4})(2.0325)^4$$

$$= 2.4140$$

which is slightly different than the value calculated in Example 1.2.1; one reason for the discrepancy is due to the Cauchy coefficients quoted in Table 1.2 being applicable over a wider wavelength range at the expense of some accuracy. Although both dispersion relations have four parameters, $A_1, A_2, \lambda_1, \lambda_2$ for Sellmeier and $n_{-2}, n_0, n_2, n_4$ for Cauchy, the functional forms are different.
Group Velocity and Group Index

• There are no perfect monochromatic waves

• We have to consider the way in which a group of waves differing slightly in wavelength travel along the z-direction
Two slightly different wavelength waves traveling in the same direction result in a wave packet that has an amplitude variation that travels at the group velocity.
\[ \omega - \delta \omega + \omega + \delta \omega \]

A periodic wave packet

\[ E_{max} \]

A single wave packet

\[ v_g = \frac{d \omega}{dk} \]
Group Velocity

Consider two sinusoidal waves that are close in frequency, that is, they have frequencies \( \omega - \delta \omega \) and \( \omega + \delta \omega \). Their wavevectors will be \( k - \delta k \) and \( k + \delta k \). The resultant wave is

\[
E_x(z,t) = E_o \cos[(\omega - \delta \omega)t - (k - \delta k)z] + E_o \cos[(\omega + \delta \omega)t - (k + \delta k)z]
\]

By using the trigonometric identity
\[
\cos A + \cos B = 2\cos[\frac{1}{2}(A - B)]\cos[\frac{1}{2}(A + B)]
\]
we arrive at

\[
E_x(z,t) = 2E_o \cos[(\delta \omega)t - (\delta k)z]\cos(\omega t - kz)
\]
This represents a sinusoidal wave of frequency $\omega$. This is amplitude modulated by a very slowly varying sinusoidal of frequency $\delta \omega$. This system of waves, i.e. the modulation, travels along $z$ at a speed determined by the modulating term, $\cos[(\delta \omega)t - (\delta k)z]$. The maximum in the field occurs when $[(\delta \omega)t - (\delta k)z] = 2m\pi = \text{constant}$ ($m$ is an integer), which travels with a velocity

$$\frac{dz}{dt} = \frac{\delta \omega}{\delta k} \quad \text{or} \quad V_g = \frac{d\omega}{dk}$$

This is the group velocity of the waves because it determines the speed of propagation of the maximum electric field along $z$. 

$$E_x(z,t) = 2E_0\cos[(\delta \omega)t - (\delta k)z]\cos(\omega t - kz)$$
The **group velocity** therefore defines the speed with which energy or information is propagated.

\[ V_g = \frac{d\omega}{dk} \]

\[ \omega = \frac{2\pi c}{\lambda_o} \text{ and } k = \frac{2\pi n}{\lambda_o}, \lambda_o \text{ is the free space wavelength.} \]

Differentiate the above equations in red

\[ d\omega = -(2\pi c/\lambda_o^2)d\lambda_o \]

\[ dk = 2\pi n(-1/\lambda_o^2)d\lambda_o + (2\pi / \lambda_o) \left( \frac{dn}{d\lambda_o} \right) d\lambda_o \]

\[ dk = -(2\pi / \lambda_o^2) \left( n - \lambda_o \frac{dn}{d\lambda_o} \right) d\lambda_o \]

\[ \therefore \quad V_g = \frac{d\omega}{dk} = \frac{-(2\pi c/\lambda_o^2)d\lambda_o}{-(2\pi / \lambda_o^2) \left( n - \lambda_o \frac{dn}{d\lambda_o} \right) d\lambda_o} = \frac{c}{n - \lambda_o \frac{dn}{d\lambda_o}} \]
where $n = n(\lambda)$ is a function of the wavelength. The group velocity $v_g$ in a medium is given by,

$$v_g(\text{medium}) = \frac{d\omega}{dk} = \frac{c}{n - \lambda \frac{dn}{d\lambda}}$$

This can be written as

$$v_g(\text{medium}) = \frac{c}{N_g}$$
Group Index

\[ N_g = n - \lambda \frac{dn}{d\lambda} \]

is defined as the \textbf{group index of the medium}.

In general, for many materials the refractive index \( n \) and hence the group index \( N_g \) depend on the wavelength of light. Such materials are called \textbf{dispersive}.
Refractive index $n$ and the group index $N_g$ of pure SiO$_2$ (silica) glass as a function of wavelength.