Examples of Fresnel Reflection and Transmission
Today’s Topics

• Examples of Fresnel Reflection and Transmission
  – Poynting vector review
  – Problem from I. Khader
  – Poynting vector example
  – Problem E Herr
  – TIR Example
  – Problem from D Miller
  – Examples from Book
Poynting’s Vector

- Field quantities (as opposed to mechanical objects) cannot be localized within a wavelength
- Following Heaviside, Pocklington and Poynting tried to track the flow of energy

\[
S(r, t) = E(r, t) \times H(r, t)
\]

\[
\nabla \cdot S(r, t) = -\frac{\partial w_e(r, t)}{\partial t} - \frac{\partial w_m(r, t)}{\partial t}
\]

\[
w_e(r, t) = \varepsilon_0 n E(r, t) \cdot E(r, t)
\]

\[
w_m(r, t) = \mu_0 H(r, t) \cdot H(r, t)
\]
Gaussian Surface and Detection

Detector surface

Far away
Poynting and Detectors (I)

\[
S(r, t) = E(r, t) \times H(r, t)
\]

\[
\int S(r, t) \cdot dA = -\frac{\partial W_e(r, t)}{\partial t} - \frac{\partial W_m(r, t)}{\partial t}
\]

\[
W_e(r, t) = \varepsilon_0 \int dV nE(r, t) \cdot E(r, t)
\]

\[
W_m(r, t) = \mu_0 \int dV H(r, t) \cdot H(r, t)
\]

- When we assume the medium is “simple” then Poynting’s theorem states that the flow of energy corresponds to a change of stored EM energy
- Currents and induced polarizations could complicate interpretation
Applying Poynting’s vector to a plane wave leads naturally to the definition of the optical intensity in terms of a time average of the fields and the wave impedance in the medium.

\[
\langle S(r, t) \rangle = \frac{1}{2} \Re \{ \mathbf{E}(r) \times \mathbf{H}^*(r) \}
\]

\[
\langle S(r, t) \rangle = -\frac{\mathbf{E}_0 \times \mathbf{k} \times \mathbf{E}_0^*}{2\omega \mu_0} \hat{e}_k
\]

\[
\langle S(r, t) \rangle = \frac{\mathbf{E}_0 \cdot \mathbf{E}_0^*}{2\eta} \hat{e}_k
\]

\[
\langle S(r, t) \rangle = I_{opt}(r, t)\hat{e}_k
\]
Poynting and Detectors (III)

- Use of a Gaussian surface for a detector leads to a natural definition of optical power collected in terms of the optical intensity.

\[ \langle S(r, t) \rangle = I_{opt}(r, t) \hat{e}_k \]

\[ \langle \langle S(r, t) \rangle \cdot dA \rangle = I_{opt}(r, t) A \hat{e}_k \cdot \hat{e}_z \]

\[ \langle S(r, t) \cdot dA \rangle = P_{opt}(r, t) \]

\[ \frac{P_{opt}(r, t)}{\cos \theta_{inc}} = \frac{1}{\tau} \int_{t-\tau}^{t} dt / dA I_{opt}(r, t) \]

- The detector time (bandwidth) shows up naturally within the time average.
Problem 2 from Problem Set I

- Solution by Isaac Khader
G. Beam in Symmetric Confocal Cavity

Kasap 1.5

a - Rayleigh range

\[ R \left( \frac{L}{2} \right) = \frac{L}{2} \left[ 1 + \left( \frac{2z_0}{L} \right)^2 \right] = L \]

\[ \implies L = 2z_0 \]

b - Beam waist

\[ z_0 = \frac{\pi w_0^2}{\lambda} \implies 2w_0 = \sqrt{\frac{4\lambda z_0}{\pi}} \quad (L=R=2z_0) \quad \sqrt{\frac{2\lambda R}{\pi}} \]

c - Plug in for 633nm light and 0.5m cavity

\[ 2w_0 \approx 0.45\text{mm} \]

\[ 2w \approx 0.63\text{mm} \]

\[ 2w = 2w_0 \sqrt{1 + \left( \frac{z}{z_0} \right)^2} \implies 2w = 2\sqrt{2}w_0 \]
Poynting Vector Example

a) Derive the time harmonic wave equation in a simple dielectric medium of index \( n \) directly from Maxwell’s equations.

b) Assuming plane wave solutions to the time harmonic wave equation, find the relationships between the wave propagation direction \( \mathbf{k} \), the electric field vector \( \mathbf{E}_0 \) and the magnetic field vector \( \mathbf{H}_0 \).

c) Find the phase velocity \( v_p \) of the vector waves in terms of the permittivity \( \varepsilon_0 \), the index \( n \) and permeability \( \mu_0 \) of the medium, where phase velocity is generally defined as \( v_p = \frac{\omega}{k} \).

d) Find the wave impedance \( \eta \), which is generally defined as the ratio of the magnitude of the electric vector \( |\mathbf{E}_0| = \sqrt{\mathbf{E}_0 \cdot \mathbf{E}_0} \) to the magnitude of the magnetic vector \( |\mathbf{H}_0| = \sqrt{\mathbf{E}_0 \cdot \mathbf{E}_0} \).

e) For an arbitrary wave vector \( \mathbf{k} \), find two polarization states. The solution to this problem is evidently not unique (why?). How could you define somewhat unique polarization states that you might be able to use in other problems.

f) Find the Poynting vector of a circularly polarized wave propagating in an arbitrary direction. Find the near field intensity of this wave as well as the power carried by the wave.
a) Derive the time harmonic wave equation in a simple dielectric medium of index $n$ directly from Maxwell’s equations.

**Solution:** We begin with time dependent Maxwell’s equations

\[
\nabla \times \mathbf{E}(r, t) = -\frac{\partial \mathbf{B}(r, t)}{\partial t}
\]

\[
\nabla \times \mathbf{H}(r, t) = \mathbf{J}(r, t) + \frac{\partial \mathbf{D}(r, t)}{\partial t}
\]

\[
\nabla \cdot \mathbf{D}(r, t) = \rho(r, t)
\]

\[
\nabla \cdot \mathbf{B}(r, t) = 0.
\]
PVE Solution II

Using the time harmonic fields

\[ E(r, t) = \Re \{ E(r)e^{-i\omega t} \} \]
\[ H(r, t) = \Re \{ H(r)e^{-i\omega t} \} \]
\[ D(r, t) = \Re \{ D(r)e^{-i\omega t} \} \]
\[ B(r, t) = \Re \{ B(r)e^{-i\omega t} \}, \]

we can write

\[ \nabla \times E(r) = i\omega B(r) \]
\[ \nabla \times H(r) = J(r) - i\omega D(r) \]
\[ \nabla \cdot D(r) = \rho(r) \]
\[ \nabla \cdot B(r) = 0. \]
PVE Solution III

The constitutive equations we want are

\[
\begin{align*}
D(\mathbf{r}) &= \varepsilon_0 n E(\mathbf{r}) \\
B(\mathbf{r}) &= \mu_0 H(\mathbf{r}) \\
J(\mathbf{r}) &= 0 \\
\rho(\mathbf{r}) &= 0
\end{align*}
\]

allowing us to obtain

\[
\begin{align*}
\nabla \times E(\mathbf{r}) &= i \omega \mu_0 H(\mathbf{r}) \\
\nabla \times H(\mathbf{r}) &= -i \omega \varepsilon_0 n E(\mathbf{r}) \\
\nabla \cdot E(\mathbf{r}) &= 0 \\
\nabla \cdot H(\mathbf{r}) &= 0.
\end{align*}
\]

Applying the curl operator, first to the electric field equation, plugging in the second and, after a vector simplification \((\nabla \times \nabla \times E) = \nabla (\nabla \cdot E) - \nabla^2 E\) using the third equation, then applying the curl operator to second equation and using the first, and then the fourth after the same vector simplification, we find

\[
\begin{align*}
\nabla^2 E(\mathbf{r}) + k^2 E(\mathbf{r}) &= 0 \\
\nabla^2 H(\mathbf{r}) + k^2 H(\mathbf{r}) &= 0 \\
k &= n \omega / c \\
c &= 1 / \sqrt{\mu_0 \varepsilon_0}
\end{align*}
\]

which are the electric and magnetic field wave equations.
b) Assuming plane wave solutions to the time harmonic wave equation, find the relationships between the wave propagation direction $\mathbf{k}$, the electric field vector $\mathbf{E}_0$ and the magnetic field vector $\mathbf{H}_0$.

**Solution:** Assuming wave equation solutions of the form

\[
\mathbf{E}(r) = \mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r}} \\
\mathbf{H}(r) = \mathbf{H}_0 e^{i\mathbf{k} \cdot \mathbf{r}}
\]

we find the relations

\[
\mathbf{k} \times \mathbf{E}_0 = -\omega \mu_0 \mathbf{H}_0 \\
\mathbf{k} \times \mathbf{H}_0 = -\omega \varepsilon_0 n^2 \mathbf{E}_0 \\
k \cdot \mathbf{E}_0 = 0 \\
k \cdot \mathbf{H}_0 = 0.
\]

c) Find the phase velocity $v_p$ of the vector waves in terms of the permittivity $\varepsilon_0$, the index $n$ and permeability $\mu_0$ of the medium, where phase velocity is generally defined as $v_p = \frac{\omega}{k}$.

**Solution:**

\[
v_p = \frac{\omega}{k} = \frac{c}{n}
\]

d) Find the wave impedance $\eta$, which is generally defined as the ratio of the magnitude of the electric vector $|\mathbf{E}_0| = \sqrt{\mathbf{E}_0 \cdot \mathbf{E}_0}$ to the magnitude of the magnetic vector $|\mathbf{H}_0| = \sqrt{\mathbf{E}_0 \cdot \mathbf{E}_0}$.
PVE Solution V

Solution:

\[ \eta = \frac{\omega \mu_0}{k} = \frac{1}{n} \sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{\omega \varepsilon_0 n^2}{k} = \frac{\eta_0}{n} \]

e) For an arbitrary wave vector \( \mathbf{k} \), find two polarization states. The solution to this problem is evidently not unique (why?). How could you define somewhat unique polarization states that you might be able to use in other problems.

**Solution:** The two states, \( \hat{\mathbf{e}}_1 \) and \( \hat{\mathbf{e}}_2 \) are defined by

\[ \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_k = \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_k = 0. \]

A possible way to decide is to choose one polarization to lie for example in the \( x-z \) plane such that it reduces to the TM state when the \( \mathbf{k} \) vector lies in the \( x-z \) plane as it does in Fresnel’s problem.

f) Find the Poynting vector of a circularly polarized wave propagating in an arbitrary direction. Find the near field intensity of this wave as well as the power carried by the wave.

**Solution:** The Poynting vector will be given by

\[ \mathbf{S}_k = I_0 \hat{\mathbf{e}}_k \]

where

\[ I_0 = \frac{\sqrt{E_0 \cdot E_0}}{2\eta}. \]

The power \( P_S \) per cross sectional area \( A \) carried across a surface with unit normal \( \hat{\mathbf{e}}_S \) will be given by

\[ \frac{P_S}{A} = I_0 \hat{\mathbf{e}}_S \cdot \hat{\mathbf{e}}_k = I_0 \cos \theta \]
Problem 3 from Problem Set 1

- Problem Solution from Alex Herr
Problem set 1.3 (Kasap 1.8)

Alec Herr

September 8, 2015
Pure silica (100% SiO$_2$)

(* Pure Silica *)

A1 = 0.696749;
A2 = 0.408218;
A3 = 0.890815;
L1 = 0.0690660;
L2 = 0.115662;
L3 = 9.900559;

n = Sqrt[1 + (A1*l^2)/(l^2-L1^2) + (A2*l^2)/(l^2-L2^2) + (A3*l^2)/(l^2-L3^2)];
Ng = n - (l*D[n,l]);

Plot[{n, Ng}, {l, 0.5, 1.8}]
Solve[D[3*10^8/Ng, l] == 0, l]
86.5% SiO₂-13.5% GeO₂

(* SiO₂ - GeO₂ *)
A₁=0.711040;
A₂=0.451885;
A₃=0.704048;
L₁=0.0642700;
L₂=0.129408;
L₃=9.425478;
n=Sqrt[1+(A₁*l^2)/(l^2-L₁^2)+(A₂*l^2)/(l^2-L₂^2)+(A₃*l^2)/(l^2-L₃^2)]
Ng=n-(l*D[n,l]);
Plot[{n,Ng},{l,0.5,1.8}]
Solve[D[Ng,l]==0,l]
Problem on TIR

Figure 1: Illustration of the geometric relationships between the field vectors and propagation directions in the Fresnel problem for (a) the TE case and (b) the TM case.
Problem on TIR II

In this problem, we want to consider the problem of incidence on air \((n_2 = n_a \approx 1.)\) from glass \((n_1 = n_g \approx 1.5)\) and, in particular, what happens when the incident angle, \(\theta_i\), in glass exceeds the TIR angle, \(\theta_{TIR} = \sin^{-1}\left(\frac{n_2}{n_1}\right)\), that is, when \(\sin \theta_i \geq 1/1.5\). Do recall that the Fresnel relations are given by

\[
\begin{align*}
    t_\perp &= \frac{2n_1 \cos \theta_i}{n_1 \cos \theta_i + n_2 \cos \theta_t} \\
    t_\parallel &= \frac{2n_1 \cos \theta_i}{n_2 \cos \theta_i + n_1 \cos \theta_t} \\
    r_\perp &= \frac{n_1 \cos \theta_t - n_2 \cos \theta_t}{n_1 \cos \theta_i + n_2 \cos \theta_t} \\
    r_\parallel &= \frac{n_2 \cos \theta_i - n_1 \cos \theta_t}{n_2 \cos \theta_i + n_1 \cos \theta_t}.
\end{align*}
\]

with

\[
\begin{align*}
    n_1 \sin \theta_i &= n_2 \sin \theta_t \\
    \theta_r &= \theta_i
\end{align*}
\]

where the situation is as pictured in Figure 1.

For purposes of normalization, we will take the incident time averaged Poynting vector to be

\[
\langle \mathbf{S}(\mathbf{r}) \rangle = \hat{\mathbf{e}}_{inc}
\]

where the incident electric field is given by

\[
\mathbf{E}(\mathbf{r}, t) = a_{TM} \cos(\mathbf{k}_{inc} \cdot \mathbf{r} - \omega t - \delta_{TM})\hat{\mathbf{e}}_{TM} + a_{TE} \cos(\mathbf{k}_{inc} \cdot \mathbf{r} - \omega t - \delta_{TE})\hat{\mathbf{e}}_{TE}
\]

where, as usual,

\[
\hat{\mathbf{e}}_{TE} = \hat{\mathbf{e}}_y
\]

and we have

\[
\hat{\mathbf{e}}_{inc} \cdot \hat{\mathbf{e}}_{TM} = \hat{\mathbf{e}}_{inc} \cdot \hat{\mathbf{e}}_{TM} = 0.
\]
Problem on TIR III

a) Find the form of transmitted and reflected fields when the incident angle exceeds the TIR angle.

b) Try to interpret the meaning of the vectors $\mathbf{k}_{ref}$ and $\mathbf{k}_t$ of these waves.

\[ \mathbf{k}_r = k (\sin \theta_i \hat{e}_x - \cos \theta_i \hat{e}_z) . \]

The $\theta_i$ is the only problem. The problem comes from

\[ \sin \theta_t = 1.5 \sin \theta_i \]
\[ \cos \theta_t = i \sqrt{(\sin^2 \theta_t - 1)} \]

such that

\[ \mathbf{k}_i = k \sin \theta_t \hat{e}_x + ik \sqrt{(\sin^2 \theta_t - 1)} \hat{e}_z . \]

To interpret, we could spend some time with trigonometric identities or use a simple time harmonic calculation and then take the real part to note that the wave propagates down the interface with and decays in the $z$ direction, a situation that is illustrated in figures 2, 3, 4. Figure 2 illustrates the usual situation when the angle of incidence is less than the TIR angle. The wavefronts are plane and the transmitted wavefront moves away from the interface. When the incident angle is near the TIR angle, the situation is as depicted in figure 3. The transmitted wave still propagates down the interface, but decays in the $z$–direction. At the TIR angle, the decay is negligible (zero), but the decay constant increases as the $\sin \theta_t$ value increases above unity. When the incident angle can be significantly greater than the TIR angle as one would like the situation to be in a waveguide, an additional feature emerges, that of the Goos–Hänchen shift as is illustrated in figure 4.

c) Find the transmitted and reflected time averaged Poynting vector’s when the incident angle exceeds the TIR angle.

d) Is any energy transmitted to coordinate $z \to \infty$ when the $\theta_i$ exceeds the TIR angle. Explain.
Figure 2: Schematic depiction of the phase fronts in the Fresnel problem. The incident (red) phase front approaches the interface from the left and the reflected phase front moves from the interface to the left. The boundary conditions require that the phases of the transmitted, reflected and incident waves line up in a quite specific way. Here we are assuming that $n_1 = n_{glass} \approx 1.5$ and $n_2 = n_{air} \approx 1.0$ as well as $\sin \theta_i \leq n_1/n_2$. In this case, the transmitted phase fronts are real so the transmitted wave is a homogeneous plane wave that is transmitted away from the interface.
Figure 3: Schematic depiction of the phase fronts in the Fresnel problem. The incident (red) phase front approaches the interface from the left and the reflected phase front moves from the interface to the left. The boundary conditions require that the phases of the transmitted, reflected and incident waves line up in a quite specific way. Here we are assuming that \( n_1 = n_{\text{glass}} \approx 1.5 \) and \( n_2 = n_{\text{air}} \approx 1.0 \) as well as \( \sin \theta_i \approx n_1/n_2 \). In this case, the transmitted phase fronts are marginally real so the transmitted wave is an almost homogeneous plane wave that is transmitted along from the interface.
a) Find the form of transmitted and reflected fields when the incident angle exceeds the TIR angle.

**Solution:** The transmitted and reflected electric fields are of the form

\[
E_r(r, t) = r_{TM} a_{TM} \cos(k_r \cdot r - \omega t - \delta_{TM}) \hat{e}_{rTM} + r_{TE} a_{TE} \cos(k_r \cdot r - \omega t - \delta_{TE}) \hat{e}_{TE}
\]

\[
E_t(r, t) = t_{TM} a_{TM} \cos(k_t \cdot r - \omega t - \delta_{TM}) \hat{e}_{tTM} + t_{TE} a_{TE} \cos(k_t \cdot r - \omega t - \delta_{TE}) \hat{e}_{TE}
\]

where the \( \hat{e}_{TM} \) has now been additionally subscripted with \( r \) or \( t \) to indicate that these vectors are dependent on the propagation direction. The \( \hat{e}_{TE} = \hat{e}_y \) is not.

Figure 4: Schematic depiction of the phase fronts in the Fresnel problem. The incident (red) phase front approaches the interface from the left and the reflected phase front moves from the interface to the left. The boundary conditions require that the phases of the transmitted, reflected and incident waves line up in a quite specific way. Here were are assuming that \( n_1 = n_{glass} \approx 1.5 \) and \( n_2 = n_{air} \approx 1.0 \) as well as \( \sin \theta_i > n_1/n_2 \). In this case, the transmitted phase fronts are complex so the transmitted wave is an inhomogeneous plane wave that is transmitted along from the interface but decays to the right of the interface.
TIR P-Solution II

c) Find the transmitted and reflected time averaged Poynting vector’s when the incident angle exceeds the TIR angle.

**Solution:** The incident, reflected and transmitted propagation complex Poynting vectors are of the form

\[
S_i = \sin \theta_i \hat{e}_x + \cos \theta_i \hat{e}_z \\
S_r = \sin \theta_i \hat{e}_x - \cos \theta_i \hat{e}_z \\
S_t = \frac{n_g}{n_a} \left( n_g \sin \theta_i \hat{e}_x - i \sqrt{\left( \left( \frac{n_1}{n_2} \right)^2 \sin^2 \theta_i - 1 \right)} \hat{e}_z \right)
\]

which is to say that the time averaged Poynting vectors are of the forms

\[
\langle S_i \rangle = \sin \theta_i \hat{e}_x + \cos \theta_i \hat{e}_z \\
\langle S_r \rangle = \sin \theta_i \hat{e}_x - \cos \theta_i \hat{e}_z \\
\langle S_t \rangle = \sin \theta_i \hat{e}_x
\]

where the \( \eta \)'s and \( n \)'s cancelled in the expression for the transmitted time averaged Poynting vector. There is no power crossing the place \( z = 0 \) here.

d) Is any energy transmitted to coordinate \( z \to \infty \) when the \( \theta_i \) exceeds the TIR angle. Explain.

**Solution:** There is no power crossing the plane \( z = 0 \) as is evident from the time averages Poynting vectors.
Problem Set 1 Problem 4

- Solution by David Miller
Problem 1.9

The index of refraction for ZeSe when $1\mu m < \lambda < 11\mu m$ can be described by the Cauchy relation:

$$n = 2.4365 + \frac{0.0485}{\lambda^2} + \frac{0.0061}{\lambda^4} - 0.0003\lambda^2$$

Find the refractive index and group index at $\lambda = 5\mu m$. 
Refractive and Group Index

Refractive index comes from plugging into the Cauchy Relation:

\[ n(5\mu m) = 2.4309 \]

The Group Index comes from the equation

\[ N_g = n - \lambda_0 \frac{dn}{d\lambda} \]

Evaluating the derivative yields:

\[ \frac{dn}{d\lambda} = -\frac{0.097}{\lambda^3} - \frac{0.0244}{\lambda^5} - 0.0006\lambda \]

Plugging in \( \lambda = 5\mu m \)

\[ N_g(5\mu m) = 2.4498 \]
Some Examples from the Book

- Internal and External Reflection
- Transmittance and Reflectance
- AR Coating
- Bragg Reflectance
Example: Reflection at normal incidence. Internal and external reflection

Consider the reflection of light at normal incidence on a boundary between a glass medium of refractive index 1.5 and air of refractive index 1.

(a) If light is traveling from air to glass, what is the reflection coefficient and the intensity of the reflected light with respect to that of the incident light?

(b) If light is traveling from glass to air, what is the reflection coefficient and the intensity of the reflected light with respect to that of the incident light?

(c) What is the polarization angle in the external reflection in a above? How would you make a polaroid from this?
Solution

(a) The light travels in air and becomes partially reflected at the surface of the glass which corresponds to external reflection. Thus $n_1 = 1$ and $n_2 = 1.5$. Then,

$$r_\parallel = r_\perp = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1 - 1.5}{1 + 1.5} = -0.2$$

This is negative which means that there is a 180° phase shift. The reflectance ($R$), which gives the fractional reflected power, is

$$R = r_\parallel^2 = 0.04 \text{ or } 4\%.$$
(b) The light travels in glass and becomes partially reflected at the glass-air interface which corresponds to internal reflection. $n_1 = 1.5$ and $n_2 = 1$. Then,

\[ r_{//} = r_{\perp} = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1.5 - 1}{1.5 + 1} = 0.2 \]

There is no phase shift. The reflectance is again 0.04 or 4%. In both cases (a) and (b) the amount of reflected light is the same.
(c) Light is traveling in air and is incident on the glass surface at the polarization angle. Here $n_1 = 1$, $n_2 = 1.5$ and $\tan \theta_p = (n_2/n_1) = 1.5$ so that $\theta_p = 56.3^\circ$.

This type of pile-of-plates polarizer was invented by Dominique F.J. Arago in 1812.
Transmittance

Transmittance $T$ relates the intensity of the transmitted wave to that of the incident wave in a similar fashion to the reflectance.

However the transmitted wave is in a different medium and further its direction with respect to the boundary is also different due to refraction.

For normal incidence, the incident and transmitted beams are normal so that the equations are simple:
Transmittance

\[ T_{\perp} = \frac{n_2 |E_{to,\perp}|^2}{n_1 |E_{io,\perp}|^2} = \left( \frac{n_2}{n_1} \right) |t_{\perp}|^2 \]

\[ T_{\parallel} = \frac{n_2 |E_{to,\parallel}|^2}{n_1 |E_{io,\parallel}|^2} = \left( \frac{n_2}{n_1} \right) |t_{\parallel}|^2 \]

or

\[ T = T_{\perp} = T_{\parallel} = \frac{4n_1 n_2}{(n_1 + n_2)^2} \]

Further, the fraction of light reflected and fraction transmitted must add to unity. Thus \( R + T = 1 \).
Reflection and Transmission – An Example

**Question**  A light beam traveling in air is incident on a glass plate of refractive index 1.50. What is the Bréster or polarization angle? What are the relative intensities of the reflected and transmitted light for the polarization perpendicular and parallel to the plane of incidence at the Brésterwer angle of incidence?

**Solution**  Light is traveling in air and is incident on the glass surface at the polarization angle $\theta_p$. Here $n_1 = 1$, $n_2 = 1.5$ and $\tan \theta_p = (n_2/n_1) = 1.5$ so that $\theta_p = 56.31^\circ$. We now have to use Fresnel’s equations to find the reflected and transmitted amplitudes. For the perpendicular polarization

$$ r_\perp = \frac{E_{r0,\perp}}{E_{i0,\perp}} = \frac{\cos \theta_i - [n^2 - \sin^2 \theta_i]^{1/2}}{\cos \theta_i + [n^2 - \sin^2 \theta_i]^{1/2}} $$

$$ r_\perp = \frac{\cos(56.31^\circ) - [1.5^2 - \sin^2 (56.31^\circ)]^{1/2}}{\cos(56.31^\circ) + [1.5^2 - \sin^2 (56.31^\circ)]^{1/2}} = -0.385 $$

On the other hand, $r_{\parallel} = 0$. The reflectances $R_\perp = |r_\perp|^2 = 0.148$ and $R_{\parallel} = |r_{\parallel}|^2 = 0$ so that $R = 0.074$, and has no parallel polarization in the plane of incidence. Notice the negative sign in $r_\perp$, which indicates a phase change of $\pi$. 
Reflection and Transmission – An Example

\[ t_\perp = \frac{E_{t0,\perp}}{E_{i0,\perp}} = \frac{2 \cos \theta_i}{\cos \theta_i + [n^2 - \sin^2 \theta_i]^{1/2}} \]

\[ t_\perp = \frac{2 \cos(56.31^\circ)}{\cos(56.31^\circ) + [1.5^2 - \sin^2(56.31^\circ)]^{1/2}} = 0.615 \]

\[ t_{\parallel} = \frac{E_{t0,\parallel}}{E_{i0,\parallel}} = \frac{2n \cos \theta_i}{n^2 \cos \theta_i + [n^2 - \sin^2 \theta_i]^{1/2}} \]

\[ t_{\parallel} = \frac{2(1.5) \cos(56.31^\circ)}{(1.5)^2 \cos(56.31^\circ) + [1.5^2 - \sin^2(56.31^\circ)]^{1/2}} = 0.667 \]

Notice that \( r_{\parallel} + nt_{\parallel} = 1 \) and \( r_\perp + 1 = t_\perp \), as we expect.
Reflection and Transmission – An Example

To find the transmittance for each polarization, we need the refraction angle $\theta_t$. From Snell's law, $n_1 \sin \theta_i = n_t \sin \theta_t$ i.e. $(1)\sin(56.31^\circ) = (1.5)\sin \theta_t$, we find $\theta_t = 33.69^\circ$.

$$T_{\parallel} = \frac{n_2 |E_{to,\parallel}|^2}{n_1 |E_{io,\parallel}|^2} = \left(\frac{n_2}{n_1}\right)|t_{\parallel}|^2$$

$$T_{\perp} = \frac{n_2 |E_{to,\perp}|^2}{n_1 |E_{io,\perp}|^2} = \left(\frac{n_2}{n_1}\right)|t_{\perp}|^2$$

$$T_{\parallel} = \left[\frac{(1.5)\cos(33.69^\circ)}{(1)\cos(56.31^\circ)}\right](0.667)^2 = 1$$

$$T_{\perp} = \left[\frac{(1.5)\cos(33.69^\circ)}{(1)\cos(56.31^\circ)}\right](0.615)^2 = 0.852$$

Clearly, light with polarization parallel to the plane of incidence has greater intensity.

If we were to reflect light from a glass plate, keeping the angle of incidence at $56.3^\circ$, then the reflected light will be polarized with an electric field component perpendicular to the plane of incidence. The transmitted light will have the field greater in the plane of incidence, that is, it will be partially polarized. By using a stack of glass plates one can increase the polarization of the transmitted light. (This type of pile-of-plates polarizer was invented by Dominique F.J. Arago in 1812.)