ECEN 5645
Introduction to Optoelectronics
Class Meeting 6

AR Coating, Bragg
Reflection and Complex Refractive Index
Today’s Topics:
Examples on

- Penetration Depth
- AR coating
- Bragg Reflection
- Complex Refractive Index
  - Reflectance
  - Example: Cd Te
Figure 1: Illustration of the geometric relationships between the field vectors and propagation directions in the Fresnel problem for (a) the TE case and (b) the TM case.
Fresnel's Equations

\[
\begin{align*}
r_\perp &= \frac{E_{r0,\perp}}{E_{i0,\perp}} = \frac{\cos \theta_i - \left[ n^2 - \sin^2 \theta_i \right]^{1/2}}{\cos \theta_i + \left[ n^2 - \sin^2 \theta_i \right]^{1/2}} \\
r_{\parallel} &= \frac{E_{r0,\parallel}}{E_{i0,\parallel}} = \frac{\left[ n^2 - \sin^2 \theta_i \right]^{1/2} - n^2 \cos \theta_i}{\left[ n^2 - \sin^2 \theta_i \right]^{1/2} + n^2 \cos \theta_i} \\
t_\perp &= \frac{E_{t0,\perp}}{E_{i0,\perp}} = \frac{2 \cos \theta_i}{\cos \theta_i + \left[ n^2 - \sin^2 \theta_i \right]^{1/2}} \\
t_{\parallel} &= \frac{E_{t0,\parallel}}{E_{i0,\parallel}} = \frac{2n \cos \theta_i}{n^2 \cos \theta_i + \left[ n^2 - \sin^2 \theta_i \right]^{1/2}}
\end{align*}
\]
Important Fresnel Phenomena

- Brewster’s Effect – no reflected wave for the TM polarization
- Total Internal Reflection – when second medium is more dense – absence of transmission
Fresnel's Equations

\[
\begin{align*}
    r_\perp &= \frac{E_{r0,\perp}}{E_{i0,\perp}} = \frac{\cos \theta_i - \left[ n^2 - \sin^2 \theta_i \right]^{1/2}}{\cos \theta_i + \left[ n^2 - \sin^2 \theta_i \right]^{1/2}} \\
    r_\parallel &= \frac{E_{r0,\parallel}}{E_{i0,\parallel}} = \frac{\left[ n^2 - \sin^2 \theta_i \right]^{1/2} - n^2 \cos \theta_i}{\left[ n^2 - \sin^2 \theta_i \right]^{1/2} + n^2 \cos \theta_i} \\
    t_\perp &= \frac{E_{t0,\perp}}{E_{i0,\perp}} = \frac{2 \cos \theta_i}{\cos \theta_i + \left[ n^2 - \sin^2 \theta_i \right]^{1/2}} \\
    t_\parallel &= \frac{E_{t0,\parallel}}{E_{i0,\parallel}} = \frac{2n \cos \theta_i}{n^2 \cos \theta_i + \left[ n^2 - \sin^2 \theta_i \right]^{1/2}}
\end{align*}
\]
Phase Shift and Penetration Depth

- TIR is used for waveguides
- TIR can be used polarization devices where the difference between phase shifts will alter reflected polarization state
- TIR is used in couplers where penetration is important
- Penetration into cladding determines loss and dispersion in propagation
Example: Reflection of light from a less dense medium (internal reflection)

A ray of light which is traveling in a glass medium of refractive index $n_1 = 1.460$ becomes incident on a less dense glass medium of refractive index $n_2 = 1.440$. The free space wavelength ($\lambda$) of the light ray is 1300 nm.

(a) What should be the minimum incidence angle for TIR?
(b) What is the phase change in the reflected wave when $\theta_i = 87^\circ$ and when $\theta_i = 90^\circ$?
(c) What is the penetration depth of the evanescent wave into medium 2 when $\theta_i = 87^\circ$ and when $\theta_i = 90^\circ$?
Solution

(a) The critical angle $\theta_c$ for TIR is given by

$$\sin \theta_c = n_2/n_1 = 1.440/1.460 \text{ so that } \theta_c = 80.51^\circ$$

(b) Since the incidence angle $\theta_i > \theta_c$ there is a phase shift in the reflected wave. The phase change in $E_{r,\perp}$ is given by $\phi_\perp$.

Using $n_1 = 1.460$, $n_2 = 1.440$ and $\theta_i = 87^\circ$, 
\[ \tan\left(\frac{1}{2} \phi_\perp\right) = \frac{\sin^2 \theta_i - n^2}{\cos \theta_i} = \frac{\sin^2 (87^\circ) - \left(\frac{1.440}{1.460}\right)^2}{\cos (87^\circ)} \]

\[ = 2.989 = \tan\left[\frac{1}{2}(143.0^\circ)\right] \]

so that the phase change \( \phi_\perp = 143^\circ \).

For the \( E_{r,\parallel} \) component, the phase change is

\[ \tan\left(\frac{1}{2} \phi_\parallel + \frac{1}{2} \pi\right) = \frac{\sin^2 \theta_i - n^2}{n^2 \cos \theta_i} = \frac{1}{n^2} \tan\left(\frac{1}{2} \phi_\perp\right) \]
so that

\[ \tan\left(\frac{1}{2}\phi_\parallel + \frac{1}{2}\pi\right) = \left(\frac{n_1}{n_2}\right)^2 \tan\left(\frac{\phi_\perp}{2}\right) = \]

\[ (1.460/1.440)^2 \tan\left(\frac{1}{2}143^\circ\right) \]

which gives \( \phi_\parallel = 143.95^\circ - 180^\circ \) or \(-36.05^\circ\)

Repeat with \( \theta_i = 90^\circ \) to find \( \phi_\perp = 180^\circ \) and \( \phi_\parallel = 0^\circ \).

Note that as long as \( \theta_i > \theta_c \), the magnitude of the reflection coefficients are unity. Only the phase changes.
(c) The amplitude of the evanescent wave as it penetrates into medium 2 is

\[ E_{t,\perp}(y,t) \propto E_{to,\perp} \exp(-\alpha_2 y) \]

The field strength drops to \( e^{-1} \) when \( y = 1/\alpha_2 = \delta \), which is called the penetration depth. The attenuation constant \( \alpha_2 \) is

\[ \alpha_2 = \frac{2\pi n_2}{\lambda} \left[ \left( \frac{n_1}{n_2} \right)^2 \sin^2 \theta_i - 1 \right]^{1/2} \]
\[
\alpha_2 = \frac{2\pi (1.440)}{(1300 \times 10^{-9} \text{ m})}\left[(\frac{1.460}{1.440})^2 \sin^2 (87^\circ) - 1\right]^{1/2}
\]

\[= 1.10 \times 10^6 \text{ m}^{-1}.\]

The penetration depth is,
\[\delta = \frac{1}{\alpha_2} = \frac{1}{(1.104 \times 10^6 \text{ m})} = 9.06 \times 10^{-7} \text{ m}, \text{ or } 0.906 \mu\text{m}\]

For 90°, repeating the calculation, \(\alpha_2 = 1.164 \times 10^6 \text{ m}^{-1}\), so that
\[\delta = \frac{1}{\alpha_2} = 0.859 \mu\text{m}\]

The penetration is greater for smaller incidence angles
Anti-Reflection Coatings

- Thin film coatings can be used to alter reflection and transmission properties by specifying thickness and index.
- Thicknesses are order of a fraction of a wavelength.
- High index contrast leads to wide bandwidth.
- Thin is relative – plasmonics and nano-optics are based on much thinner coatings.
Example: Antireflection coatings on solar cells

When light is incident on the surface of a semiconductor it becomes partially reflected. Partial reflection is an important energy loss in solar cells.

The refractive index of Si is about 3.5 at wavelengths around 700 - 800 nm. Reflectance with $n_1(\text{air}) = 1$ and $n_2(\text{Si}) \approx 3.5$ is

$$R = \left( \frac{n_1 - n_2}{n_1 + n_2} \right)^2 = \left( \frac{1 - 3.5}{1 + 3.5} \right)^2 = 0.309$$
30% of the light is reflected and is not available for conversion to electrical energy; a considerable reduction in the efficiency of the solar cell.

Illustration of how an antireflection coating reduces the reflected light intensity.
We can coat the surface of the semiconductor device with a thin layer of a dielectric material, *e.g.* Si$_3$N$_4$ (silicon nitride) that has an intermediate refractive index.

\[ n_1(\text{air}) = 1, \quad n_2(\text{coating}) \approx 1.9 \quad \text{and} \quad n_3(\text{Si}) = 3.5 \]

Light is first incident on the air/coating surface. Some of it becomes reflected as $A$ in the figure. Wave $A$ has experienced a $180^\circ$ phase change on reflection because this is an external reflection. The wave that enters and travels in the coating then becomes reflected at the coating/semiconductor surface.
This reflected wave $B$, also suffers a 180° phase change since $n_3 > n_2$.

When $B$ reaches $A$, it has suffered a total delay of traversing the thickness $d$ of the coating twice. The phase difference is equivalent to $k_c(2d)$ where $k_c = 2\pi / \lambda_c$ is the propagation constant in the coating, i.e. $k_c = 2\pi / \lambda_c$ where $\lambda_c$ is the wavelength in the coating.

Since $\lambda_c = \lambda / n_2$, where $\lambda$ is the free-space wavelength, the phase difference $\Delta \phi$ between $A$ and $B$ is $(2\pi n_2 / \lambda)(2d)$. To reduce the reflected light, $A$ and $B$ must interfere destructively. This requires the phase difference to be $\pi$ or odd-multiples of $\pi$, $m\pi$ where $m = 1, 3, 5, \ldots$ is an odd-integer. Thus
\[
\left(\frac{2\pi n_2}{\lambda}\right) 2d = m\pi
\]

or

\[
d = m\left(\frac{\lambda}{4n_2}\right)
\]

The thickness of the coating must be **odd-multiples** of the quarter wavelength in the coating and depends on the wavelength.

\[
R_{\text{min}} = \left(\frac{n_2^2 - n_1 n_3}{n_2^2 + n_1 n_3}\right)^2
\]
To obtain good destructive interference between waves $A$ and $B$, the two amplitudes must be comparable. We need (proved later) $n_2 = \sqrt{n_1 n_3}$. When $n_2 = \sqrt{n_1 n_3}$ then the reflection coefficient between the air and coating is equal to that between the coating and the semiconductor. For a Si solar cell, $\sqrt{3.5}$ or 1.87. Thus, Si$_3$N$_4$ is a good choice as an antireflection coating material on Si solar cells.

Taking the wavelength to be 700 nm,

$$d = m \left( \frac{\lambda}{4n_2} \right)$$

where $d$ is the thickness of the coating, $\lambda$ is the wavelength, $m$ is an integer, and $n_2$ is the refractive index of the coating. For a wavelength of 700 nm and a refractive index of 1.9,

$$d = \frac{700 \text{ nm}}{4 \times 1.9} = 92.1 \text{ nm} \text{ or odd-multiples of } d.$$
\[ R_{\text{min}} = \left( \frac{n_2^2 - n_1 n_3}{n_2^2 + n_1 n_3} \right)^2 \]

\[ R_{\text{min}} = \left( \frac{1.9^2 - (1)(3.5)}{1.9^2 + (1)(3.5)} \right)^2 = 0.00024 \text{ or } 0.24\% \]

Reflection is almost entirely extinguished
However, only at 700 nm.
Problem Set 2 Problem 1 (1.10 Kasap)

- Solution by Evolene Premillieu
Problem 1.10 : Refractive index, reflection, and the Brewster’s angle

Light of $\lambda_0=1300$ nm traveling in pure Silica medium.

I used Figure 1.8 : Refractive index $n$ and the group index $N_g$ of pure Silica $\text{SiO}_2$ (glass) as a function of wavelength.

$n=1.4457 \quad N_g=1.462$ and the velocity of light $c=3.10^8 \text{ m.s}^{-1}$

The phase velocity is given by : $v=\frac{c}{n}=2.075.108 \text{ m/s}$
The group velocity is given by : $v_g=\frac{c}{N_g}=2.052.108 \text{ m/s}$

As $N_g > n$, the group velocity is smaller than the phase velocity.

Light traveling in Silica is incident on a Silica-air interface :

Brewster’s angle is given by :

$$\tan \theta_p = n_2/n_1 \quad \theta_p=34.67^\circ$$

Critical angle for Total Internal Reflection is given by :

$$\sin \theta_c = n_2/n_1 \quad \theta_c=43.76^\circ$$

At the polarization angle the light is linearly polarized as the electric field oscillations are contained within a well-defined plane, perpendicular to the plane of incidence and the direction of propagation.

Light beam traveling in Silica, incident at normal incidence on a Silica-air interface :

Reflection coefficient : $r=n_1-n_2/n_1+n_2 =0.1822$ \hspace{1cm} Reflectance : $R=r^2=0.0332$ or 3.32%

Light beam traveling in Silica, incident at normal incidence on an air-Silica interface :

Reflection coefficient : $r=n_1-n_2/n_1+n_2 = -0.1822$ \hspace{1cm} Reflectance : $R=r^2=0.0332$ or 3.32% => there is a $\pi$ phase shift.
Bragg Reflectors

• AR coatings are designed such that reflections from successive layers cancel

• Reflective coatings are designed such that reflections from successive layers constructively interfere

• A high low reflective structure will require a different design procedure than a low high

• Each layer of a Bragg reflector is lossless, that is, the overall loss is zero, that is, \( R + T = 1 \)
Dielectric Mirror or Bragg Reflector

Schematic illustration of the principle of the dielectric mirror with many low and high refractive index layers

\[ n_0 \rightarrow n_1 \rightarrow n_2 \rightarrow n_1 \rightarrow n_2 \rightarrow \text{Substrate} \]

\[ N = 1 \rightarrow N = 2 \]
Dielectric mirrors

Schematic illustration of the principle of the dielectric mirror with many low and high refractive index layers
A **dielectric mirror** has a stack of dielectric layers of alternating refractive indices. Let $n_1 (= n_H) > n_2 (= n_L)$

Layer thickness $d = \text{Quarter of wavelength or } \lambda_{\text{layer}}/4$

$\lambda_{\text{layer}} = \lambda_o/n$; $\lambda_o$ is the free space wavelength at which the mirror is required to reflect the incident light, $n = \text{refractive index of layer}$.

Reflected waves from the interfaces interfere constructively and give rise to a substantial reflected light. If there are sufficient number of layers, the reflectance can approach unity at $\lambda_o$. 
$r_{12}$ for light in layer 1 being reflected at the 1-2 boundary is

$$r_{12} = \frac{n_1 - n_2}{n_1 + n_2}$$

and is a positive number indicating no phase change.

$r_{21}$ for light in layer 2 being reflected at the 2-1 boundary is

$$r_{21} = \frac{n_2 - n_1}{n_2 + n_1}$$

which is $-r_{12}$ (negative) indicating a $\pi$ phase change.

The reflection coefficient alternates in sign through the mirror.

The phase difference between A and B is

$$0 + \pi + 2k_1d_1 = 0 + \pi + 2(2\pi n_1/\lambda_0)(\lambda_0/4n_1) = 2\pi.$$

Thus, waves A and B are in phase and interfere constructively.

Dielectric mirrors are widely used in modern vertical cavity surface emitting semiconductor lasers.
Dielectric Mirror or Bragg Reflector

**Δλ** = Reflectance bandwidth (Stop-band for transmittance)
Consider an “infinite stack”
For reflection, the phase difference between \( A \) and \( B \) must be

\[
2k_1d_1 + 2k_2d_2 = m(2\pi)
\]

\[
2(2\pi n_1/\lambda)d_1 + 2(2\pi n_2/\lambda)d_2 = m(2\pi)
\]

\[
n_1d_1 + n_2d_2 = \frac{m\lambda}{2}
\]
Dielectric Mirror or Bragg Reflector

\[ n_1 d_1 + n_2 d_2 = \lambda / 2 \]

\[ d_1 = \lambda / 4 n_1 \]

\[ d_2 = \lambda / 4 n_2 \]

Quarter-Wave Stack

\[ d_1 = \lambda / 4 n_1 \text{ and } d_2 = \lambda / 4 n_2 \]
Dielectric Mirror or Bragg Reflector

\[ R_N = \left[ \frac{n_1^{2N} - (n_0 / n_3)n_2^{2N}}{n_1^{2N} + (n_0 / n_3)n_2^{2N}} \right]^2 \]

\[ \frac{\Delta \lambda}{\lambda_o} \approx \frac{4}{\pi} \arcsin \left( \frac{n_1 - n_2}{n_1 + n_2} \right) \]
Problem Set 2 Problem 2 (1.14 Kasap)

- Solution by Fermat - to be presented by Jeremy Shugrue
Complex Refractive Index

\[ \alpha = -\frac{dI}{Idz} \]
Complex Index

- Materials are made of atoms
- Electrons in atoms, bonds between atoms, molecular bonds have stationary states
- Transitions to higher states eventually decay and lead to absorption bands
- Absorption appears as an imaginary part of the dielectric constant
- Kramer’s Kronig gives the real part of any causal function in terms of the imaginary
Complex Refractive Index

Consider $k = k' - jk''$

$$E = E_o \exp(-k''z) \exp(j(\omega t - k' z))$$

$$I \propto |E|^2 \propto \exp(-2k''z)$$

We know from EM wave theory

$$\varepsilon_r = \varepsilon_r' - j\varepsilon_r'' \quad \text{and} \quad N = \varepsilon_r^{1/2}$$

$$N = n - jK = k/k_o = (1/k_o)[k' - jk'']$$

$$N = n - jK = \sqrt{\varepsilon_r} = \sqrt{\varepsilon_r' - j\varepsilon_r''}$$
Reflectance

\[ \varepsilon_r = \varepsilon_r' - j \varepsilon_r'' \]
and
\[ N = \varepsilon_r^{1/2} \]
\[ N = n - jK \]
\[ n^2 - K^2 = \varepsilon_r' \]
and
\[ 2nK = \varepsilon_r'' \]

\[ R = \left| \frac{n - jK - 1}{n - jK + 1} \right|^2 = \frac{(n - 1)^2 + K^2}{(n + 1)^2 + K^2} \]
CdTe is used in various applications such as lenses, wedges, prisms, beam splitters, antireflection coatings, windows *etc* operating typically in the infrared region up to 25 µm. It is used as an optical material for low power CO$_2$ laser applications.
Complex Refractive Index

\[ N = n - jK = \sqrt{\varepsilon_r} = \sqrt{\varepsilon'_r - j\varepsilon''_r} \]

\[ n^2 - K^2 = \varepsilon'_r \quad \text{and} \quad 2nK = \varepsilon''_r \]

\[ R = \frac{(n - jK - 1)^2}{(n - jK + 1)} = \frac{(n - 1)^2 + K^2}{(n + 1)^2 + K^2} \]

88 µm
Example: Complex Refractive Index for CdTe

Calculate the absorption coefficient $\alpha$ and the reflectance $R$ of CdTe at the Reststrahlen peak, and also at 50 µm. What is your conclusion?

Solution: At the Reststrahlen peak, $\lambda \approx 70$ µm, $K \approx 6$, and $n \approx 4$. The free-space propagation constant is

$$k_0 = \frac{2\pi}{\lambda} = \frac{2\pi}{(70 \times 10^{-6} \text{ m})} = 9.0 \times 10^4 \text{ m}^{-1}$$

The absorption coefficient $\alpha$ is $2k$,

$$\alpha = 2k'' = 2k_0 K = 2(9.0 \times 10^4 \text{ m}^{-1})(6) = 1.08 \times 10^6 \text{ m}^{-1}$$

which corresponds to an absorption depth $1/\alpha$ of about 0.93 micron.
Solution continued: At the Reststrahlen peak, $\lambda \approx 70 \ \mu m$, $K \approx 6$, and $n \approx 4$, so that

$$R = \frac{(n-1)^2 + K^2}{(n+1)^2 + K^2} = \frac{(4-1)^2 + 6^2}{(4+1)^2 + 6^2} \approx 0.74 \text{ or } 74\%$$

At $\lambda = 50 \ \mu m$, $K \approx 0.02$, and $n \approx 2$. Repeating the above calculations we get

$$\alpha = 5.0 \times 10^3 \text{ m}^{-1}$$

$$R = 0.11 \text{ or } 11\%$$

There is a sharp increase in the reflectance from 11 to 72% as we approach the Reststrahlen peak.