Gaussian Beam Wave Equation

\[ \nabla^2 u + k^2 u = 0 \]

In the paraxial regime, slowly varying envelope

\[ u(r) = a(r)e^{-ikz} = |a(r)| \cos[2\pi n t - k z + \angle a(r)] \]

\[ \Rightarrow a(r) \text{ must satisfy} \]

\[ \left( \frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} + \frac{\partial^2 a}{\partial z^2} \right) e^{-ikz} - i2ke^{-ikz} \frac{\partial a}{\partial z} - k^2e^{-ikz}a + k^2e^{-ikz}a = 0 \]

Paraxial Helmholtz

\[ \nabla^2 u - i2k \frac{\partial u}{\partial z} = 0 \]

Paraxial approx of spherical wave \( u(r) = \frac{1}{r}e^{-ikr} \)

soln Paraxial Wave

\[ a(r) = \frac{A_1}{z} e^{-i\frac{k^2}{2z}} \quad \rho^2 = x^2 + y^2 \]

Gaussian Beam Intensity

\[ I(r, z) = |U(r, z)|^2 = I_0 \left[ \frac{w_0^2}{w(z)} \right]^2 e^{-2r^2/w_0^2} \]

Far-field diffraction angle

\[ \theta_{beam} = \tan^{-1} \left( \frac{\lambda}{\pi w_0 \eta(z)} \right) \quad z \gg z_0 \]

Paraxial Gaussian Field

\[ E(x, y, z) = E_0 \frac{\omega_0}{\omega(z)} e^{-i[kz - \eta(z)] - r^2/(2w(z)^2)} \]

\[ \eta(z) = \tan^{-1} \left( \frac{z}{z_0} \right) \quad \text{phase factor} \]

\[ R(z) = z \left( 1 + \frac{z}{z_0} \right) \quad \text{Radius of Curvature} \]

\[ \omega^2(z) = \omega_0^2 \left( 1 + \frac{z}{z_0} \right) \quad \text{beam size} \]

Phase along PQ is shorter than PR \( \Rightarrow \) phase \( \eta \)

On axis peak intensity

\[ I(0, z) = I_0 \left[ \frac{w_0}{w(z)} \right]^2 = \frac{I_0}{1 + (z/z_0)^2} \]

\[ |z| \gg z_0 \]

\[ I(0, z) \approx I_0 \frac{w_0^2}{z_0^2} \quad \text{inverse square law} \]
Beam radius

\[ w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_0}\right)^2} \]

beam intensity drops to \(1/e^2 = 0.135\) at \(w(z)\)

At \(z = 0\) minimum radius = \(w_0\)

\(2w_0 = d_0=\) spot size = diameter

\(z \gg z_0\)

\[ w(z) \approx w_0 \frac{z}{z_0} = \theta_0 z \]

\[ \theta_0 = \frac{w_0}{z_0} = \frac{\sqrt{\lambda}}{\pi z_0} \]

Divergence half angle

\[ \theta_0 = \frac{\lambda}{\pi w_0} \]

Full angle in terms of spot diameter \(d_0\)

\[ \theta_f = 2\theta_0 = \frac{2\lambda}{\pi d_0} \]

\[ \frac{2\pi w_0^2}{\lambda} = \frac{\pi d_0^2}{2} \]

Ex \(\lambda = 633\)

\(2w_0 = 2\text{cm} \quad \Rightarrow \quad 2z_0 \approx 1\text{km}\)

\(2w_0 = 2\text{mm} \quad \Rightarrow \quad 2z_0 \approx 10\text{m}\)

\(2w_0 = 0.2\text{mm} \quad \Rightarrow \quad 2z_0 \approx 10\text{cm}\)

\(2w_0 = 20\mu\text{m} \quad \Rightarrow \quad 2z_0 \approx 1\text{mm}\)

Depth of focus

within a distance \(z_0\) of waist

\(w(z)\) grows to \(\sqrt{2}w_0\)

Area doubles, peak on axis power halves

Depth of focus \(\equiv 2\times\) Rayleigh range

Wavefront Curvature

Surface of Constant phase (q=integer)

\[ k \left[z + \frac{\rho^2}{2R(z)}\right] - \eta(z) = 2\pi q \]

\[ z + \frac{\rho^2}{2R(z)} = q\lambda + \frac{\eta \lambda}{2\pi} \]

Radius of curvature \(R\)

\[ R(z) = z \left[1 + \left(\frac{z_0}{z}\right)^2\right] = \frac{z^2 + z_0^2}{z} \]

Minimum ROC where derivative =0

\[ R'(z) = \frac{z(2z) - (z^2 + z_0)}{z^2} = \frac{z^2 - z_0^2}{z^2} = 0 \quad \Rightarrow \quad z = z_0 \]

Highest curvature spherical wave at \(\pm z_0\) away from waist

Guoy Phase

\[ \eta(z) = \tan^{-1}\left(\frac{z}{z_0}\right) \]

Phase along PQ is shorter than PR \(\rightarrow\) phase \(\eta\)
ABCD law

Arbitrary paraxial system characterized by ABCD matrix \( M \)

\[
\begin{bmatrix}
  y_x \\
  u_x
\end{bmatrix} = \begin{bmatrix}
  A & B \\
  C & D
\end{bmatrix} \begin{bmatrix}
  y_i \\
  u_i
\end{bmatrix}  \quad |M| = AD - BC = 1
\]

Propagation of Gaussian beam \( q \) parameter governed by

\[
q_2 = \frac{A_1q_1 + B_1}{C_1q_1 + D_1}
\]

Cascade

\[
q_2 = \frac{A_2A_1q_1 + B_2}{C_2A_1q_1 + D_2C_1q_1 + D_2} = \frac{A_1q_1 + B_1}{C_1q_1 + D_1} + \frac{A_2q_1 + B_2}{C_2q_1 + D_2}
\]

and using ABCD matrix multiplication

\[
\begin{bmatrix}
  A & B \\
  C & D
\end{bmatrix} = \begin{bmatrix}
  A_2 & B_2 \\
  C_2 & D_2
\end{bmatrix} \begin{bmatrix}
  A_1 & B_1 \\
  C_1 & D_1
\end{bmatrix} = \begin{bmatrix}
  A_2A_1 + B_2C_1 & A_2B_1 + B_2D_1 \\
  C_2A_1 + D_2C_1 & C_2B_1 + D_2D_1
\end{bmatrix}
\]

Lens Focal Spot Shift

Gaussian laser beam incident with its waist at the lens has \( \infty \) radius of curvature.

Just after the lens, radius of curvature \( = f \)

\[
R(z) = z \left[ 1 + \left( \frac{z_0}{z} \right)^2 \right] = f
\]

where \( z \) is measured from the waist and is negative at lens.

This will focus with a new waist at a position where \( R(z) = 0 \), which gives a shift of the Gaussian focus away from the geometrical focus

\[
z = f - \frac{z_0^2}{2} \approx \frac{z_0^2}{2f}
\]

where the approximate form is since in this geometry \( z \approx f \)

Gaussian Beam Thru Lens

Newtonian lens formula

\(-xx' = f^2\)

\(-x = z - f\)

\(x' = z' - f\)

Gaussian Beam Lens Formula adds a new term to account for diffraction

\((z - f)(z' - f) = f^2 - f_0^2\)

where

\[
\begin{align*}
f_0^2 &= z_0^2 z_0' = 4\lambda \frac{\pi}{\theta_f^2} \frac{\pi}{\theta_f'^2} = \frac{\pi d_0^2}{4\lambda} \frac{\pi d_0'^2}{4\lambda} = \frac{\pi^2 d_0^2}{4\lambda^2} \frac{\pi^2 d_0'^2}{4\lambda^2} = \frac{\pi^2 d_0^2}{4\lambda^2} \frac{\pi^2 d_0'^2}{4\lambda^2} = \frac{4\lambda}{\pi \theta_f \theta_f'} = \frac{4\lambda}{\pi \theta_f' \theta_f}.
\end{align*}
\]

where

\[
\begin{align*}
&u_0 = \frac{d_0}{\theta_0} \
&\frac{d_0}{\theta_0} = \frac{d_0}{\theta_0'} = \frac{\pi d_0 d_0'}{4\lambda} = \frac{\pi w_0 w_0'}{\lambda} = \frac{\lambda}{4\lambda} \frac{4\lambda}{\pi \theta_f \theta_f'} = \frac{4\lambda}{\pi \theta_f' \theta_f}.
\end{align*}
\]
Complex ABCD matrices

Consider soft Gaussian transmissive aperture
\[
t(y) = \frac{E_i(y)}{E_0(y)} = e^{-\alpha y^2/2} = e^{-y^2/2\sigma^2}
\]
Incident plane wave becomes Gaussian
Incident centered Gaussian beam is modified
\[
E_i(y) = e^{-\frac{k^2y^2}{2\sigma^2}} = t(y)E_0(y) = e^{-\frac{y^2}{2}\frac{k^2}{\sigma^2}}
\]
\[
\frac{1}{q_2(z)} = \frac{1}{q_1(z)} - \frac{i\alpha}{k^2} - \frac{1}{q_1(z)} - \frac{i}{k\sigma^2}
\]
Thus
\[
q_2 = \frac{q_1}{-i\epsilon_0 + 1} \frac{Aq_1 + B}{Cq_1 + D}
\]
Gives an effective ABCD matrix much like a lens, but with a complex part
\[
\begin{bmatrix}
1 & 0 \\
-i\epsilon_0 & 1
\end{bmatrix}
\]

Geometrical interpretation of Gaussian Beam Radius of Curvature

Equation of circle passing through two foci at (x1(x1,r1) and (−x1,−r1) and with center (c,0)(C,0) is
\[
x^2 + y^2 - 2cx = 0
\]
Need to show that the radius of this circle is half the radius of curvature for a given axis intercept point z.

Family of circles pivoted at chord locations (0, ±z0) determine curvature of Gaussian wavefronts as well as waist 2w0 = \(\sqrt{\frac{4\lambda z_0}{\pi}}\).

Gaussian Beam radius of curvature
\[
R(z) = z \left(1 + \frac{k^2}{4z^2}\right)
\]
Cord of semilength z0, center at d
\[
d^2 = r^2 - z_0^2 \quad \Rightarrow \quad d = \sqrt{r^2 - z_0^2}
\]
distance along axis to intersection
\[
z = d + r \quad \Rightarrow \quad z - r = d = \sqrt{r^2 - z_0^2}
\]
\[
z^2 - 2rz = r^2 = r^2 - z_0^2
\]
Solving for the radius of this circle
\[
r = -\frac{1}{2z}(z + z_0) = \frac{z}{2} \left[1 + \frac{z_0}{z^2}\right] = \frac{R(z)}{2}
\]
Solid circle radius is half radius of curvature of Gaussian beam (indicated by dotted circle) with Rayleigh range z0 = half of chord for any axis intersection distance z.

Gaussian Beams refracting into medium of index n

Geometrical determination of resonator beam parameters and stability

Intersection of two circles tangent to the mirror surfaces and with half the radius of the mirrors curvature locates the waist at the chord of length \(d_{chord} = 2z_0\), and determines the size of the beamwaist and divergence angle as
\[
2w_0 = \sqrt{\frac{2\lambda z_0}{\pi}} \quad 2\theta_0 = 2\sqrt{\frac{2\lambda}{\pi z_0}}
\]
Collin’s diagram

$q$ is the complex radius of curvature parameter. Lens law gives change of radius of curvature $\frac{1}{R_2} - \frac{1}{R_1} = \frac{1}{f}$ giving jump

$$\frac{1}{q_1} - \frac{1}{q_2} = \frac{1}{f} \quad \Rightarrow \quad \frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{f}$$

Free space propagation through $z$ gives

$$q_2 = q_1 + z$$

Plotting this trajectory in the $j/q$ plane gives straight vertical lines going to $\infty$ along the real axis while plotting in the $j/q$ plane gives circles of radius $1/z_0 = \lambda/\pi w_0^2$

Complex beam parameter

remember $q_0$ is pure imaginary

$$\frac{1}{q(z)} = \frac{1}{z + q_0 z + q_0^2} = \frac{z - q_0}{z^2 + |q_0|^2} = \frac{1}{R(z)} + i \frac{\lambda}{\pi w^2(z)}$$

Radius of curvature extracted from real part of $q(z)$

$$R(z) = \frac{z^2 + \left(\frac{\pi w_0^2}{\lambda z}\right)^2}{z} = z \left[1 + \left(\frac{\pi w_0^2}{\lambda z}\right)^2\right]$$

Beam radius extracted from imaginary part of $q(z)$

$$w^2(z) = \frac{\lambda}{\pi} \left[2 + \left(\frac{\pi w_0^2}{\lambda z}\right)^2\right] = w_0^2 \left[1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2\right]$$

$$\frac{q}{q_0(z)} = A e^{-i\phi} \quad \Rightarrow \quad \phi = \tan^{-1} \frac{z\lambda}{\pi w_0^2}$$

Gaussian Beams using Shamir’s Operators

$$u_{i}(x, y) = e^{-\rho^2/w_0^2} = e^{\frac{i k}{2} \left(\frac{x^2}{2w_0^2} + \frac{y^2}{2w_0^2}\right)} = Q_{[1/q_0]}$$

Imaginary radius of curvature

$$\frac{1}{q_0} = \frac{2i}{k w_0^2} = \frac{i \lambda}{\pi w_0^2}$$

Free space propagation of Gaussian

$$z = \frac{i \lambda d}{Q_{[1/q_0]^2}} + z \frac{q_0}{q_0 + z} Q_{[1/q_0] \left(\frac{1}{w^2} \right)} = e^{\frac{i k z}{\lambda}} Q_{[1/q_0]}$$

Using canonic operator for the case $D \neq 0$

$$T u_1 = \mathcal{R}_{[B/D]} \left\{ \mathcal{A}_{[C/D]} Q_{[1/q]} \right\} = \mathcal{R}_{[B/D]} \left\{ Q_{[D]} \right\} \mathcal{A}_{[C/D]} \left\{ Q_{[1]} \right\}$$

Followed by FPO by $B/D$ and using $AD - BC = 1 \Rightarrow BC = AD - 1$

$$q_{out} = q' + \frac{B}{D} C q + D = \frac{D (C q + D)}{D (C q + D)} q + (AD - \lambda) q + BD = A q + B C q + D$$

Thus

$$u_{out} = \frac{q q_{out}}{q_{out} Q_{[1/q_{out}]}} = \frac{\mathcal{A}_{[C/D]} \left\{ Q_{[1]} \right\}}{A q + B C q + D} Q_{[1]} = \frac{q}{D(A q + B) Q_{[1/q_0]}}$$