Gaussian Beam Wave Equation

Helmholtz eqn
\[ \nabla^2 u + k^2 u = 0 \]

In the paraxial regime, slowly varying envelope
\[ u(r) = a(r)e^{-ikz} = |a(r)| \cos[2\pi vt - k z + \angle a(r)] \]

⇒ \( a(r) \) must satisfy
\[ \left( \frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} + \frac{\partial^2 a}{\partial z^2} \right) e^{-ikz} - i2ke^{-ikz} \frac{\partial a}{\partial z} - k^2 e^{-ikz} a + k^2 e^{-ikz} a = 0 \]

Paraxial Helmholtz
\[ \nabla^2 a - i2k \frac{\partial a}{\partial z} = 0 \]

Paraxial approx of spherical wave \( u(r) = \frac{1}{z} e^{-ikz} \)

soln Paraxial Wave
\[ a(r) = \frac{A_1}{z} e^{-ik\rho^2} \quad \rho^2 = x^2 + y^2 \]

Gaussian Beam Solution

Shift the origin of this soln \( z \to z - \xi \)
\[ a(r) = \frac{A_1}{q(z)} e^{-ik\rho^2} \quad q(z) = z - \xi \]

\( \xi \) complex, still a soln, let \( \xi = -iz_0 \) where \( z_0 \) will turn out to be the Rayleigh range
\[ a(r) = \frac{A_1}{q(z)} e^{-ik\rho^2} \quad q(z) = z + iz_0 \]

Separate \( \frac{1}{q(z)} \) into its real and imaginary parts
\[ \frac{1}{q(z)} = R(z) e^{i\pi/4} \quad \frac{1}{z + iz_0} = \frac{z}{z^2 + z_0^2} \quad \frac{1}{z - iz_0} = \frac{z}{z^2 + z_0^2} \]

Radius \( R(z) = z \left( 1 + \frac{z_0^2}{z^2} \right) \)

beam waist \( w^2(z) = \frac{\lambda}{2\pi} \frac{1}{z_0} \left( 1 + \frac{2z_0^2}{z^2} \right) \)

Additional phase of \( \frac{A_1}{q(z)} = A_2 \frac{z - iz_0}{z + iz_0} = A_2 \frac{z - iz_0}{\frac{z^2}{z_0^2}} \quad w_0^2 \)
\[ \eta(z) = \tan^{-1} \left( \frac{-z_0}{z} \right) \]

Gaussian Beams

Rayleigh range: beam area expands \( \times 2 \)
\( z_0 = \pi\omega_0^2 n/\lambda \)

Far-field diffraction angle
\[ \theta_{\text{beam}} = \tan^{-1} \left( \frac{\lambda}{\pi\omega_0 y_i} \right) \quad z \gg z_0 \]

Paraxial Gaussian Field
\[ E(x, y, z) = E_0 \frac{\omega_0}{\omega(z)} e^{-i\left[ k(z - \eta(z)) - \rho^2 \left( \frac{1}{w_0^2} + \frac{1}{w(z)^2} \right) \right]} \]
\[ \eta(z) = \tan^{-1} \left( \frac{z}{z_0} \right) \quad \text{phase factor} \]
\[ R(z) = z \left( 1 + \frac{z_0^2}{z^2} \right) \quad \text{Radius of Curvature} \]
\[ \omega^2(z) = \omega_0^2 \left( 1 + \frac{z_0^2}{z^2} \right) \quad \text{beam size} \]

Phase along PQ is shorter than PR \( \to \) phase \( \eta \)

Gaussian Beam Intensity

\[ I(r, z) = |U(r, z)|^2 = I_0 \left[ \frac{w_0}{w(z)} \right]^2 e^{-\frac{2\pi^2}{\lambda^2}} \]

On axis peak intensity
\[ I(0, z) = I_0 \left[ \frac{w_0}{w(z)} \right]^2 = \frac{I_0}{1 + (z/z_0)^2} \]

\( |z| \gg z_0 \)
\[ I(0, z) \approx I_0 \frac{z_0^2}{z^2} \quad \text{inverse square law} \]
\[ w(z) = w_0 \sqrt{1 + \left( \frac{z}{z_0} \right)^2} \quad w_0 = \sqrt{\frac{\lambda z_0}{\pi}} \]

beam intensity drops to \(1/e^2 = 0.135\) at \(w(z)\)

At \(z = 0\) minimum radius \(w_0\)

\[ 2w_0 = d_0 \Rightarrow \text{spot size = diameter} \]

\[ z \gg z_0 \]

\[ w(z) \approx w_0 \frac{z}{z_0} = \theta_0 z \]

Divergence half angle

\[ \theta_0 = \frac{\lambda}{\pi w_0} \]

Full angle in terms of spot diameter \(d_0\)

\[ \theta_f = 2\theta_0 = \frac{2\lambda}{\pi d_0/2} = \frac{4\lambda}{\pi d_0} \]

\[ 2z_0 = \frac{\pi w_0^2}{\lambda} = \frac{\pi d_0^2}{2} \lambda \]

\[ \text{Ex } \lambda = 633 \]

\[ 2w_0 = 2\text{cm} \quad \Rightarrow \quad 2z_0 \approx 1\text{km} \]

\[ 2w_0 = 2\text{mm} \quad \Rightarrow \quad 2z_0 \approx 10\text{m} \]

\[ 2w_0 = 0.2\text{mm} \quad \Rightarrow \quad 2z_0 \approx 10\text{cm} \]

\[ 2w_0 = 20\mu\text{m} \quad \Rightarrow \quad 2z_0 \approx 1\text{mm} \]

\[ \eta(z) = \tan^{-1} \left( \frac{z}{z_0} \right) \]

Phase factor

Phase along PQ is shorter than PR \(\Rightarrow\) phase \(\eta\)
**ABCD law**

Arbitrary paraxial system characterized by ABCD matrix \( M \)

\[
\begin{bmatrix}
  y_2 \\ u_2
\end{bmatrix} = \begin{bmatrix}
  A & B \\ C & D
\end{bmatrix} \begin{bmatrix}
  y_1 \\ u_1
\end{bmatrix}
\]

\(|M| = AD - BC = 1\)

Propagation of Gaussian beam \( q \) parameter governed by

\[ q_2 = \frac{Aq_1 + B}{Cq_1 + D} = \left( \frac{1}{R} - i \frac{\lambda}{\pi nw^2} \right)^{-1} \]

Cascade

\[ q_2 = \frac{A_1q_1 + B_1}{C_1q_1 + D_1} \]

\[ q_3 = \frac{A_2A_1 + B_2}{C_2A_1 + D_2} = \frac{(C_2A_1 + D_2C_1)q_1 + (A_2B_1 + B_2D_1)}{(C_2A_1 + D_2C_1)q_1 + (C_2B_1 + D_2D_1)} = \frac{Aq_1 + B}{Cq_1 + D} \]

and using ABCD matrix multiplication

\[
\begin{bmatrix}
  A & B \\ C & D
\end{bmatrix} = \begin{bmatrix}
  A_2 & B_2 \\ C_2 & D_2
\end{bmatrix} \begin{bmatrix}
  A_1 & B_1 \\ C_1 & D_1
\end{bmatrix} = \begin{bmatrix}
  A_2A_1 + B_2C_1 & A_2B_1 + B_2D_1 \\ C_2A_1 + D_2C_1 & C_2B_1 + D_2D_1
\end{bmatrix}
\]

**Lens Focal Spot Shift**

Gaussian laser beam incident with its waist at the lens has \( \infty \) radius of curvature. Just after the lens, radius of curvature \( = f \)

\[
R(z) = z \left[ 1 + \left( \frac{2z}{z} \right)^2 \right] = f
\]

where \( z \) is measured from the waist and is negative at lens

This will focus with a new waist at a position where \( R(z) = 0 \), which gives a shift of the Gaussian focus away from the geometrical focus

\[
z - f = \frac{\lambda}{z} \approx \frac{\lambda}{f}
\]

where the approximate form is since in this geometry \( z \approx f \)

**Gaussian Beam Thru Lens**

Newtonian lens formula

\[-xx' = f^2 \]

\[-x = z - f \]

\[x' = z' - f \]

Gaussian Beam Lens Formula adds a new term to account for diffraction

\[(z - f)(z' - f) = f^2 - f_0^2\]

where

\[
f_0^2 = z_0z'_0 = \frac{4\lambda}{\pi \omega_0^2 \pi \omega_{0'}^2} = \frac{\pi \omega_0^2 \pi \omega_{0'}^2}{4\lambda} = \frac{\pi \omega_0^2}{4\lambda} = \frac{\lambda}{2\pi \theta_f' \theta_f'}
\]

\[
f_0 = \frac{w_0}{\theta_0} = \frac{d_0}{\theta_f} = \frac{d_0}{\theta_{0'}'} = \frac{\pi \omega_0 d_0}{4\lambda} = \frac{\pi \omega_{0'} d_0'}{4\lambda} = \frac{\lambda}{2\pi \theta_f \theta_f'}
\]
Complex ABCD matrices

Consider soft Gaussian transmissive aperture
\[ t(y) = \frac{E_i(y)}{E_i(y)} = e^{-\alpha y^2/2} = e^{-\gamma^2/2\sigma^2} \]

Incident plane wave becomes Gaussian
Incident centered Gaussian beam is modified
\[ E_i(y) = e^{-\frac{k y^2}{2\sigma^2}} \]
\[ t(y) E_i(y) = e^{\frac{-k y^2}{2\sigma^2}} \cdot e^{\frac{-k y^2}{2\sigma^2}} \]
\[ \frac{1}{q_2(z)} = \frac{1}{q_1(z)} - \frac{2a}{k2} = \frac{1}{q_1(z)} - \frac{a}{k} = \frac{1}{q_1(z)} - \frac{i}{k\sigma^2} \]

Thus
\[ q_2 = \frac{q_1}{i\sigma q_1 + 1} = \frac{Aq_1 + B}{Cq_1 + D} \]

Gives an effective ABCD matrix much like a lens, but with a complex part
\[ \begin{bmatrix} 1 & 0 \\ i\sigma & 1 \end{bmatrix} \]

Geometrical interpretation of Gaussian Beam

Radius of Curvature

Gaussian Beam radius of curvature
\[ R(z) = z \left[ 1 + \frac{z_0^2}{z^2} \right] \]

Cord of semilength \( z_0 \), center at \( d \)
\[ d^2 = r^2 - z_0^2 \Rightarrow d = \sqrt{r^2 - z_0^2} \]

distance along axis to intersection
\[ z = d + r \Rightarrow z - r = d = \sqrt{r^2 - z_0^2} \]

\[ z^2 - 2zr + r^2 - r^2 - z_0^2 \]

Solving for the radius of this circle
\[ r = \frac{1}{2z} (z^2 + z_0^2) = \frac{z}{2} \left[ 1 + \frac{z_0^2}{z^2} \right] = \frac{R(z)}{2} \]

Interception of two circles tangent to the mirror surfaces and with half the radius of the mirrors curvature locates the waist as at the chord of length \( d_{chord} = 2z_0 \), and determines the size of the beam waist and divergence angle as
\[ 2w_0 = \frac{\sqrt{2\lambda z_0}}{\pi} \]
\[ 2\theta_0 = \frac{\sqrt{2\lambda}}{\pi z_0} \]
Collin’s diagram

$q$ is the complex radius of curvature parameter. Lens law gives change of radius of curvature \( \frac{1}{R_1} - \frac{1}{R_2} = \frac{1}{f} \), giving jump

\[
\frac{1}{q_1} - \frac{1}{q_2} = \frac{1}{f}
\Rightarrow
\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{f}
\]

Free space propagation through $z$ gives

\[ q_2 = q_1 + z \]

Plotting this trajectory in the $j\phi$ plane gives straight vertical lines going to $\infty$ along the real axis while plotting in the $j/q$ plane gives circles of radius $1/z_0 = \lambda/\pi w_0^2$

Gaussian Beams using Shamir's Operators

\[
u_i(x,y) = e^{-x^2/z_0^2} = e^{i\frac{\lambda}{(\pi z_0)^2}} = Q_{[1/z_0]}
\]

Imaginary radius of curvature

\[ \frac{1}{q_0} = \frac{2i}{\lambda z_0} = \frac{i\lambda}{\pi w_0^2} \]

Free space propagation of Gaussian

\[ R_{[z]}(u_i) = \frac{e^{i\lambda z}}{i\lambda z Q_{[1/z]}(x,y) Q_{[1/z]}(x,y)} \left\{ \mathcal{F} \left\{ Q_{[1/z]}(x,y) Q_{[1/z]}(x,y) \right\} \right\} \]

\[ \mathcal{F} \left\{ Q_{[1/z]} \right\} = i\lambda Q_{[-\lambda^2 z]} \]

\[ \frac{e^{i\lambda z} i\lambda z q_0}{q_0 + z} Q_{[1/z]} \left[ \frac{Q_{[1/z]}(\lambda z q_0 - j\lambda z)}{Q_{[1/z]}(\lambda z q_0 + j\lambda z)} \right] = e^{i\lambda z} \frac{q_0}{q_0 + z} Q_{[1/z]} \left[ \frac{Q_{[1/z]}(\lambda z q_0 - j\lambda z)}{Q_{[1/z]}(\lambda z q_0 + j\lambda z)} \right] \]

Complex beam parameter

Remember $q_0$ is pure imaginary

\[
\frac{1}{q(z)} = \frac{1}{z + q_0 z + q_0^2} = \frac{z - q_0}{z^2 + |q_0|^2} = \frac{1}{R(z)} + i\frac{\lambda}{\pi w^2(z)}
\]

Radius of curvature extracted from real part of $q(z)$

\[ R(z) = \frac{z^2 + \left( \frac{\pi w^2}{\lambda z} \right)^2}{z} = z \left[ 1 + \left( \frac{\pi w^2}{\lambda z} \right)^2 \right] \]

Beam radius extracted from imaginary part of $q(z)$

\[ w^2(z) = \frac{\lambda}{\pi} \left( \frac{\pi w^2}{\lambda z} \right)^2 = w_0^2 \left[ 1 + \left( \frac{\lambda z}{\pi w^2} \right)^2 \right] \]

\[ \frac{q}{q_0(z)} = Ae^{-i\phi} \]

\[ A = \frac{w_0}{w(z)} \]

\[ \tan \phi = \frac{z}{\pi w_0^2 / \lambda} \Rightarrow \phi = \tan^{-1} \frac{z \lambda}{\pi w_0^2} \]

ABCD Propagation of Gaussian Beams using Operators

Using canonical operator for the case $D \neq 0$

\[ T_{u_1} = R_{[\pi / D]} \left\{ \mathcal{F} \left\{ Q_{[\pi / D]} Q_{[1/z]} \right\} \right\} = R_{[\pi / D]} \left\{ Q_{[\pi / D]} Q_{[1/z]} \right\} \]

Gaussian beam $q' = \frac{q}{D(zCq + D)}$

Followed by FPO by $B/D$ and using $AD - BC = 1$ gives

\[ q_{out} = q' + \frac{Bq}{D(Cq + D)} + \frac{q + Bq(Cq + D)}{D(Cq + D)} = q + (AD - 1)q + BD \]

\[ = Aq + B \]

\[ = \frac{q}{D(Cq + D)} \]

Thus

\[ u_{out} = \frac{q'}{q_{out}} Q_{[1/z_{out}]} = \frac{q}{D(Aq + B) Q_{[1/z_{out}]}^2} = \frac{q}{D(Aq + B) Q_{[1/z_{out}]}} \]
**4F imaging**

Object \[ \rightarrow \text{FT Lens} \rightarrow \text{FT plane} \rightarrow \text{FT Lens} \rightarrow \text{Output Image} \]

\[
T_1 = \frac{e^{i2\pi f_1}}{i\lambda f_1} \mathcal{F}\{h(x,y)\}
\]

\[
T = T_1 T_2 = \frac{e^{i2\pi f_2}}{i\lambda f_2} \mathcal{F}\{h(x,y)\}
\]

Scaled inverted imaging with no quadratic phase factor.

**Afocal telescopic imaging system**

**Impulse Response**

**Input a delta function**

\[ t(x,y) = \delta(x-x_0, y-y_0) \]

Geometrical Optics Image

\[ u_G(x,y) = \frac{1}{|M|} \delta\left(\frac{x}{|M|} + x_0, \frac{y}{|M|} + y_0\right) \]

Suppose \( M = -2 \) \( \Rightarrow \frac{x}{2} = -x_0 \Rightarrow x = -2x_0, y = -2y_0 \)

Circular Aperture Stop

\[ P(x', y') = \text{circ} \left[ \frac{r'}{R} \right] \quad r' = \sqrt{x'^2 + y'^2} \]

**Impulse Response**

\[ \rho = \sqrt{u^2 + v^2} \]

\[ h(x,y) = \mathcal{F}^{-1} \left\{ \mathcal{F}^{-1} \left\{ P(-u\lambda f_2, -v\lambda f_2) \right\} \right\} = \mathcal{F}^{-1} \left\{ \text{circ} \left[ \frac{r' \lambda f_2}{R} \right] \right\} = \frac{R}{\lambda f_2} J_1(2\pi \rho R/\lambda f_2) / \rho \]

Output

\[ u_3(x,y) = \frac{1}{|M|} \delta\left(\frac{x}{|M|} + x_0, \frac{y}{|M|} + y_0\right) \ast \ast\left[ \frac{R}{\lambda f_2} \right]^2 \text{jinc} \left( \frac{2\rho R}{\lambda f_2} \right) \]

1st zero \( 2\rho R/\lambda f_2 = 1.22 \) \( \Rightarrow \rho = 1.22\lambda f_2/D \)

**4-F Telescopic Imaging System with an aperture**

\[ a_3(x,y) = a_2(x,y) \ast h(x,y) \]

Cascaded Fourier transform modules with pupil \( P(x',y') \)

\[ a_2(x',y') = \mathcal{F}\{a_1(\xi,\eta)\} = A_1 \left( \frac{x'}{\lambda f_1}, \frac{y'}{\lambda f_1} \right) \]

\[ a_3(x,y) = \mathcal{F}\{a_2(x',y')P(x',y')\} = \frac{1}{(f_2)^2} a_1 \left( -\frac{x}{\lambda f_2}, -\frac{y}{\lambda f_2} \right) \ast P\left( \frac{x}{\lambda f_2}, \frac{y}{\lambda f_2} \right) \]

Geometrical optics image with \( M = -\frac{f_2}{f_1} \)

\[ a_2(x,y) = \frac{1}{|M|} a_1 \left( \frac{x}{M}, \frac{y}{M} \right) = \frac{f_1}{f_2} a_1 \left( -\frac{xf_1}{f_2}, -\frac{yf_1}{f_2} \right) \]

Coherent Impulse Response (CIR)

\[ h(x,y) = \frac{1}{(\lambda f_2)^2} \mathcal{F}\left\{ P(-u\lambda f_2, -v\lambda f_2) \right\} H(u,v) = P(-u\lambda f_2, -v\lambda f_2) \]

Coherent Transfer Function