Vander Lugt Complex Spatial Filter

Want to form a correlation integral

\[ a(x', y') = \int \int g(x, y) h^*(x - x', y - y') \, dx \, dy = \mathcal{F}^{-1} \{ G(u, v) H^*(u, v) \} \]

Need to perform a product of transforms. However, need to represent complex information in the transform plane. Use holography.

\[ r(x', y') = r e^{i \beta \phi r^2 \sin \theta + \frac{\alpha}{2} r^2 \sin \theta} \]

\[ a(x', y') = \frac{1}{\lambda F} \int \int h(x, y) e^{\frac{2\pi i}{\lambda F} (x' x + y' y)} \, dx \, dy = \frac{1}{i \lambda F} \mathcal{F} \left( \frac{x'}{\lambda F}, \frac{y'}{\lambda F} \right) H \left( \frac{x'}{\lambda F}, \frac{y'}{\lambda F} \right) \]

Vander Lugt Complex Spatial Filter Exposure

The amplitude transmission of the mask is proportional to intensity and exposure time

\[ t(x', y') = \kappa T_0 |a(x', y')|^2 \]

\[ t(x', y') = \kappa T_0 \left[ \frac{1}{\lambda^2 F^2} |H(u, v)|^2 + r^2 \right] \]

\[ + \frac{1}{i \lambda F} H \left( \frac{x'}{\lambda F}, \frac{y'}{\lambda F} \right) r e^{-2 \pi \alpha u \lambda F} \]

\[ + \frac{1}{i \lambda F} H^* \left( \frac{x'}{\lambda F}, \frac{y'}{\lambda F} \right) r e^{2 \pi \alpha u \lambda F} \]

\[ = \kappa T_0 \left[ \frac{1}{\lambda^2 F^2} |H(u, v)|^2 + |r|^2 + \frac{2}{\lambda F} \pi |H(u, v)| \cos \left( 2 \pi \alpha u \lambda F - \angle H(u, v) \right) \right] \]

Develop and reposition the filter.

Vander Lugt Complex Spatial Filter Readout

Transmission of the field amplitude through the hologram

\[ b(x', y') = \frac{1}{i \lambda F} G \left( \frac{x'}{\lambda F}, \frac{y'}{\lambda F} \right) \]

\[ b(u, v) = \frac{1}{i \lambda F} G(u, v) \]

\[ d(u, v) = b(u, v) t(u, v) = \frac{1}{i \lambda F} G(u, v) \left[ \kappa T_0 \left( \frac{1}{\lambda^2 F^2} |H(u, v)|^2 + |r|^2 \right) \right. \]

\[ + \frac{1}{i \lambda F} H(u, v) r e^{-2 \pi \alpha u \lambda F} + \frac{i}{\lambda F} H^*(u, v) r e^{2 \pi \alpha u \lambda F} \left. \right] \]

\[ = \kappa T_0 \frac{1}{i \lambda F} \left[ G |r|^2 + \frac{G |H|^2}{\lambda^2 F^2} + \frac{G H^*}{i \lambda F} e^{-2 \pi \alpha u \lambda F} + \frac{G H^*}{-i \lambda F} e^{2 \pi \alpha u \lambda F} \right] \]

Vander Lugt Complex Spatial Filter Output Plane

The final lens Fourier transforms this amplitude distribution. (Note the coordinate inversion)

\[ D(x'', y'') = \frac{\kappa T_0}{-\lambda^2 F^2} \left[ r^2 g(x'', y'') + \frac{1}{\lambda^2 F^2} \right] h(x'', y'') \ast h^*(-x'', -y'') \]

\[ + \frac{r}{i \lambda F} g(x'', y'') \ast h(x'', y'') \ast \delta(x - \alpha \lambda F) \]

\[ + \frac{r}{-i \lambda F} g(x'', y'') \ast h(-x'', -y'') \ast \delta(x + \alpha \lambda F) \]

Remember, that this is an alternative representation of a correlation

\[ g(x'', y'') \ast h^*(-x'', -y'') = \int \int g(x, y) h^* ((x'' - x), (y'' - y)) \, dx \, dy \]

\[ = \int \int g(x, y) h^* (x - x'', y - y'') \, dx \, dy = g(x'', y'') \ast h(x'', y'') \]

* represents convolution

* represents correlation
Why Correlators for Image Pattern Recognition?

Given a signal \( s(x, y) \) of unknown location, \((x_0, y_0)\), buried in additive noise, \(n(x, y)\).

Input

\[ f(x, y) = s(x - x_0, y - y_0) + n(x, y) \]

Shift invariant problem \(\Rightarrow\) Correlation with impulse response \(h(x, y)\)

Maximize ratio of peak signal energy in correlation peak to mean square noise energy

\[ \max \{SNR\} = \frac{\int \int |s(x, y)|^2 dxdy}{\int \int |n(x, y)|^2 dxdy} \]

Peak signal

\[ g = \int \int s(x, y)h(x - x_0, y - y_0) dx dy \]

Mean Square Noise – white noise with PSD \(N_0\)

\[ \sigma^2 = E\left\{ \int \int n(x - x_0, y - y_0) n(x - x_0, y - y_0) dx dy \right\} = \frac{N_0}{2} \int \int \delta(x - x_0) \delta(y - y_0) dx dy = \frac{1}{2} \int \int h^2(x_0, y_0) dx dy \]

\[ SNR = \frac{g^2}{\sigma^2} = \left( \frac{\int \int |s(x, y)|^2 h(x, y) dx dy}{\frac{1}{2} \int \int h^2(x_0, y_0) dx dy} \right)^2 \]

Pattern classification

\[
\begin{aligned}
\text{Sensor} & \quad \text{Feature Extraction} & \quad \text{Classifier} \quad \text{decision}
\end{aligned}
\]

Conditional Gaussian density

\[ p(z|w) = \frac{1}{(2\pi)^{D/2} |\Sigma|^{1/2}} \exp\left\{ -\frac{1}{2} (z - \mu)^T \Sigma^{-1} (z - \mu) \right\} \]

Covariance matrix \(\Sigma = E\{(z - \mu)(z - \mu)^T\}\), and class mean \(\mu_i\).

Simplest case: Independent features, same variance, Gaussian, i.e., pixels in an image \(z = (z_1, \ldots, z_D)^T\), Noise clusters are hyperspheres around means \(\mu_i\).

Discriminant function for the \(i\)th class, pick the largest \textit{a posteriori} probability

\[ g_i(z) = p(w_i|z)P(w_i) = \frac{p(z|w_i)p(w_i)}{p(z)} \]

\[ p(z) = \sum_i p(z|w_i)p(w_i) \]

Linear Pattern classification

\[ g_i(z) = \log p(z|w_i) + \log P(w_i) - \log p(z) \]

\[ = -\frac{|z - \mu_i|^2}{2\sigma^2} + \log P(w_i) - \frac{1}{2} \log |\Sigma_i| - \frac{1}{2} \log 2\pi \]

\[ = -\frac{(z - \mu_i)^T(z - \mu_i)}{2\sigma^2} + \log P(w_i) \]

\[ = -\frac{1}{2\sigma^2} \left( z^T z - 2z^T \mu_i + \mu_i^T \mu_i \right) + \log P(w_i) \]

Which is a minimum distance classifier

Quadratic term \(z^T z\) is the same for all \(g_i\).

Correlation Detection (LDF)

\[ g_i(z) = w_i^T z + w_0 \]

\[ w_i = \frac{\mu_i}{\sigma^2} \]

\[ w_0 = -\frac{1}{\sigma^2} \mu_i + \log P(w_i) \]

When \(\frac{1}{\sigma^2} \mu_i^2 \gg \sigma^2\) the class probabilities \(P(w_i)\) has small effect.

If a priori's are class independent then \(P(w_i) = P(1)\) for all \(i\) so can dropped \(P(w_i)\)
Fisher's Linear Discriminant

Projection onto a line

\[ y = w \cdot x \]

Projection of the means

\[ m_y = w \cdot m_x \]

Separation of the means

\[ |m_1 - m_2| = |w \cdot (m_1 - m_2)| \]

Scatter of projected samples

\[ s_y^2 = \sum_{x \in X} (y - m_y)^2 \]

Two class total scatter is \( s_1^2 + s_2^2 \).

Fisher's discriminant maximizes the ratio of the separation of the means to the total scatter

\[ \max_w J(w) = \max_w \frac{|m_1 - m_2|}{s_1^2 + s_2^2} = w \]

Rotated objects in Fourier Pattern recognition

Collimated Beam  FT Lens  Filter  FT Lens  Output

\[ f(x, y) \rightarrow F'_{(f, f')} \]


Scatter Matrices

Within Class Scatter Matrix

\[ S_W = \sum_{x \in X} (x - m_x)(x - m_x)^t \]

\[ S_W = \sum_{x \in X} w(x - m_x)(x - m_x)^t = w S_W w \]

Between Class Scatter Matrix

\[ S_B = (m_1 - m_2)(m_1 - m_2)^t \]

\[ (m_1 - m_2)^2 = (w^t m_1 - w^t m_2)^2 = w^t S_B w \]

\[ \rightarrow J(w) = \frac{w^t S_B w}{w^t S_W w} \]

is the generalize Rayleigh quotient.

Weight vector \( w \) that maximizes \( J \) must satisfy

\[ S_B w = \lambda S_W w \rightarrow S_B w \parallel (m_1 - m_2) \]

This gives the weight vector that defines the linear function with maximum ratio of between class to within class scatter.

Recognizing Diatoms

Recognizing Diatoms

FIG. 2.4-4. Horseshoe-simulated input of diatoms 1–6 (upper left) used to make matched filters. Correlation signals are shown for entire diatom set (top row), also as a function rotation to about 45 deg. (After Fujii and Almeida [44].)

Recognizing Shapes and Edges

Recognizing Binary Image Patterns
Recognizing Faces: Between Class and Within Class Variations

- Rotation and Scale changes decrease correlation peak
- Build Invariance by correlating against library of rotated and scaled prototypes
- Average filters use average across invariance class
- Too much averaging destroys recognition and discrimination

Recognizing Faces: Edge Enhancement to Enhance Discrimination

- Poor discrimination
- Poor recognition
- Useless to average
- Edge enhance

Discrimination and Invariance in Optical Correlation

Correlation Peak results across invariance classes
**Edge Enhanced Optical Correlation**

- Edge enhance with DC block in Fourier plane
- Excellent recognition and discrimination
- Edge enhanced prototypes can be averaged across invariance class
- Edge enhancing with a DC block is required for Fourier correlators to be useful

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**Vander Lugt Optical Correlator Processing Power**

Correlation for pattern recognition

\[ g(x', y') = \kappa \int \int f(x, y) h^*(x + x', y + y') \, dx \, dy \]

2-D Processing power: \(10^3 \times 10^3\) pixel correlation in 10ms.

Throughput and accuracy limited by detector and SLM frame rate (10ms)

\[
(10^3)^2 / 10^{-3} \text{s} = 10^6 \text{ analog multiplies per second}
\]

FFT computational equivalent throughput for \(N = n \times n\) pixel image

\[ 6N \log N = 6n^2 \log n = 20 \times 10^6 / 10^{-3} \text{s} = 12 \text{ GOP} \]

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**Optical Correlators for Pattern Recognition**

- Match the physics to the problem
- Matched spatial filter optimizes SNR
- Holographic recording of filter (VanDer Lugt Correlator 1963)
- Processed high-res spy satellite data
- 1000s of times faster than available techniques at the time
- Invention of the FFT in 1965 changed computational complexity from \(N^2\) to \(N \log N\) (up to \(n^2 \log n\) for \(n \times n\) image)
- Fastest possible Fourier operation limited by speed of light
- Most systems are input-output bottleneck limited (SLMs, CCDs, hologram, film)

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**Edge Enhanced Optical Correlation Peaks across invariance class**

- Good discrimination
- Good recognition
- Average filters work
- \(2N\) average filters vs \(N^2\) prototype filters
Synthetic Discriminant Functions – SDF

Given a set of training images, \( f_n(x, y) \), belonging to the desired class and centered. Choose the filter as a linear combination

\[
h(x, y) = \sum_n a_n f_n(x, y) \quad \text{and} \quad \mathbf{h} = \sum_n a_n \mathbf{f}_n
\]

For Equal Correlation Peak (ECP) filters, require

\[
\int \int h(x, y) f_m(x, y) \, dx \, dy = 1 \quad \forall m
\]

\[
\mathbf{h} \cdot \mathbf{f}_1 = 1
\]

Correlation matrix

\[
r_{mn} = \frac{f_m \cdot f_n}{\mathbf{R}} \quad \text{N x N matrix}
\]

\[
\mathbf{f}_n \cdot \sum_k a_k \mathbf{f}_k = \sum_k a_k \mathbf{f}_n \cdot \mathbf{f}_k = \sum_k a_k r_{nk} = 1
\]

\[
\mathbf{R} = \mathbf{u} u^T
\]

where \( \mathbf{u} \) is a vector of ones \( \mathbf{u} = [1, 1, 1, 1, 1, \ldots, 1] \)

**Soln**

\[
\mathbf{a} = \mathbf{R}^{-1} \mathbf{u}
\]

\[
\mathbf{h} = \sum_n \left( \mathbf{R}^{-1} \mathbf{u} \right) \cdot \mathbf{f}_n
\]

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Mellin Transform for scale invariant pattern recognition

\[
M(i \omega_x, i \omega_y) = \int_\infty f(x, y) x^{i \omega_x - 1} y^{i \omega_y - 1} \, dx \, dy
\]

Consider FT of \( f(e^x) \) and substitute \( \xi = \ln x, x = e^\xi \)

\[
M(i \omega_x, i \omega_y) = \int_\infty f(e^\xi, e^\eta) e^{-i(\omega_x \xi + \omega_y \eta)} \, d\xi \, d\eta = \int_\infty f(e^\xi, e^\eta) e^{-i\omega_x \xi} e^{-i\omega_y \eta} \, d\xi \, d\eta
\]

\[
= \int_\infty f(x, y) e^{-i\omega_x \ln x} e^{-i\omega_y \ln y} \, dx \, dy = \int_\infty f(x, y) x^{-i\omega_x - 1} y^{-i\omega_y - 1} \, dx \, dy
\]

Now scale the function (must pre-center it about the origin first) and substitute \( x = ax' \)

\[
\int_0^\infty f(ax', ay') x^{-\omega_x - 1} y^{-\omega_y - 1} \, dx' \, dy' = \int_0^\infty f(x, y) \frac{x^{-\omega_x - 1} y^{-\omega_y - 1} \, dx \, dy}{a^2}
\]

\[
= a^{-\omega_x} a^{-\omega_y} \int_0^\infty f(x, y) x^{-\omega_x - 1} y^{-\omega_y - 1} \, dx \, dy = a^{-\omega_x} a^{-\omega_y} M(i \omega_x, i \omega_y)
\]

Which is a phase shifted Mellin transform, so mod squaring produces an identical result.

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Interclass SDF

given \( N \) training images in \( M \) classes, \( N_1 + N_2 + \ldots + N_M = N \).

\[
\mathbf{h}_j \cdot \mathbf{h}_k = \delta_{jk}
\]

Where each of \( h_j \) is a linear combination

\[
\mathbf{h}_j = \sum_n a_j^nf_n
\]

Solution for \( a_j^i \)

\[
\mathbf{a}^j = \mathbf{R}_j^{-1} \mathbf{u}^j
\]

\( \mathbf{R}_j \) is \( N \times N \) correlation matrix for \( f_j \)

\[
\begin{align*}
\mathbf{f}_j \in \text{class A} & \quad \mathbf{f}_k \in \text{class B} \\
\mathbf{h}_j \cdot \mathbf{h}_k = 1 & \quad \mathbf{h}_j \cdot \mathbf{h}_k = 0
\end{align*}
\]

**Joint Transform Correlator \( ^a \)**

\[
t(u, v) = \kappa \int \int [f(x, y - b) + h(x, y + b)] e^{-2\pi \iota (u x + v y)} \, dx \, dy
\]

\[
= |F(u, v) e^{-2\pi \iota v b} + H(u, v) e^{+2\pi \iota v b}|^2
\]

\[
= |F|^2 + |H|^2 + F^* H e^{+2\pi \iota v b} + H^* F e^{-2\pi \iota v b}
\]

Develop and reilluminate transparancy with plane wave, FT transform diffracted wave:

\[
a(x, y) = \int_\infty t(u, v) e^{-2\pi \iota (ux + vy)} \, du \, dv = f * h * h * h * h * \delta(y + 2b) + h * f * \delta(y - 2b)
\]
Joint Transform Correlator

- Large processing gain, extracts target from noise
- Looking for peak gives good detection probability
- Wavefront computing collapses to answer
- Single photon would perform computation, but with noisy probability of detection

Incoherent Optical Correlators 1: Raster Scanning Correlator

- Transparency placed on a two axis motorized stage (or scanning with galvo-mirrors or AO deflectors)
- All of the transmitted light is collected and integrated on a wide area detector.

\[ c(x', y') = \iint |f(x, y)|^2 |h(x - x', y - y')|^2 dx \, dy \]

Motion in \( x' = vt \) is at continuous velocity \( v \) followed by a step \( \Delta y \) in \( y \) after each end of scan reset each \( X = \nu T \). Resulting raster scan of correlation

\[ c(mT + t) = \iint |f(x, y)|^2 |h(x - vt, y - m\Delta y)|^2 dx \, dy \]

Shadow Casting Correlator

Based on geometrical optics. Light as rays, breaks down with diffraction. Thus limited to \( 100 \times 100 \) resolution, so not for high resolution images. Instead used in cellular logic arrays.

Limited to positive Point Spread Functions (PSF).

Point source creates a PSF by casting a shadow of the aperture. Multiple point sources add up to give the correlation.

\[ g(x_0, y_0) = \iint f \left( \frac{x - x_0}{b}, \frac{y - y_0}{b} \right) |h(x, y)|^2 dx \, dy \]

\( f(x, y) \) could be a self luminous CRT, or uniformly illuminated ground glass and transparency, or a diffuse source and transparency.

Shadow Casting Correlator

Light at source position \((-x_0, -y_0)\) is collimated, modulated by mask \( f(x, y) \) propagating as a plane wave at an angle \( \tan \theta_x \approx \theta_x = x_0/f \) and \( \tan \theta_y \approx \theta_y = y_0/f \).

After propagating \( d \) and striking mask \( h(x, y) \) the 2 images will be shifted by \( \frac{d \tan \theta_x}{f} \) and \( \frac{d \tan \theta_y}{f} \).

So the product that is integrated by the final lens to the focal position \((x_0, y_0)\).

\[ I(x_0 = x_0, y_0 = y_0) = \kappa \iint f \left( \frac{x - d}{f x_0}, \frac{y - d}{f y_0} \right) |h(x, y)|^2 dx \, dy \]